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Abstract

Given a multivariate polynomial $F(x, y, \ldots, z)$, this paper deals with calculating the roots of F w.r.t. x in terms of formal power series or fractional-power series in $y, \ldots, z$. If the problem is regular, i.e. the expansion point is not a singular point of a root, then the calculation is easy, and the irregular case is considered in this paper. We extend the generalized Hensel construction slightly so that it can be applied to the irregular case. This extension allows us to calculate the roots of bivariate polynomial $F(x, y)$ in terms of Puiseux series in y. For multivariate polynomial $F(x, y, \ldots, z)$, we consider expanding the roots into fractional-power series w.r.t. the total-degree of $y, \ldots, z$, and the roots are expressed in terms of the roots of much simpler polynomials.

Running head: Solving Multivariate Algebraic Equation

1 Introduction

Given a multivariate polynomial $F(x, y, \ldots, z)$, this paper deals with calculating the roots of $F(x, y, \ldots, z)$, with respect to x, in terms of formal power series or fractional-power series in $y, \ldots, z$, up to any given finite power.

This calculation is fundamentally important in mathematics and mathematical sciences, see [Wal91] for a survey of recent studies. For bivariate polynomial $F(x, y)$, a simple method called Newton-Puiseux's method has been discovered in 1850, and we have already an elegant theory of Puiseux series (fractional-power series) of algebraic functions; see [P1850] and [Wal78].

Classical Newton-Puiseux's method is, however, very inefficient in that i) we must introduce many algebraic numbers practically and ii) we must perform power series substitution and solve an equation at the determination of each expansion coefficient. Hence, the classical method contains many problems for the practical use. As for i), Duval [Duv89] for example presents a method which decreases the number of algebraic numbers to be introduced. As for ii), Kung and Traub [KT78] proposes to use Newton's iterative method for power series [GCL92]. These studies have made the computation of power series roots practical, so long as bivariate polynomials are concerned. However, we have many problems in multivariate case.

Suppose the expansion point of the power series is $(y, \ldots, z) = (0, \ldots, 0)$, and let the roots of $F(x, 0, \ldots, 0)$ be $\zeta_1, \ldots, \zeta_d$:

$$F(x, 0, \ldots, 0) = (x - \zeta_1) \cdots (x - \zeta_d). \tag{1.1}$$

If we are allowed to compute numbers $\zeta_1, \ldots, \zeta_d$ numerically then the computation and the resulting expressions of the roots will be much simpler than treating them as algebraic numbers. In this paper, we admit both approaches: $\zeta_1, \ldots, \zeta_d$ may be computed numerically hence approximately to any desired accuracy or may be treated exactly as algebraic numbers. Another important note is that the computation is crucially dependent on whether or not the following condition is satisfied.

Condition A *The numbers $\zeta_1, \ldots, \zeta_d$ are mutually different, i.e.,*

$$\zeta_i \neq \zeta_j \quad \text{for any} \quad i \neq j. \tag{1.2}$$

Following [KT78], we say that the problem is *regular* if Condition A is satisfied. Note that if $\zeta_i = \zeta_j$ $(i \neq j)$ then ζ_i is a singular point of a root of $F(x, y, \ldots, z)$. If the problem is regular, the computation of the roots is nothing but the Taylor series expansion and quite easy: we have only to apply power series Newton's iterative method or perform the generalized Hensel construction, as will be explained in **2**. For irregular problems, however, both Newton's iterative method and the generalized Hensel construction breaks down. For bivariate polynomials, Kung and Traub proposed a transformation which converts irregular problems to regular ones. However, the transformation cannot directly be applied to multivariate case, and Condition A poses us a problem. In this paper, we propose a new method which is applicable to not only bivariate but also multivariate polynomials, by extending the generalized Hensel construction.

There is a close relationship between Newton's iterative method and the method using Hensel construction. That is, the method using Hensel construction is a parallel execution of linearly convergent Newton's method, calculating all the roots simultaneously; see [Wan79] or [SS92] for the parallel Hensel construction. Furthermore, there is a good similarity between calculating the roots of univariate polynomial $f(x)$ numerically and expanding the roots of multivariate polynomial $F(x, y, \ldots, z)$ symbolically. Single-root Newton's method for solving $f(x) = 0$ numerically, with initial approximation $x = x^{(0)}$ such that $f(x^{(0)}) \simeq 0$, calculates the root by increasing the accuracy iteratively as $x^{(k+1)} = x^{(k)} + \Delta x^{(k+1)},\ k = 0, 1, \ldots$, such that $|\Delta x^{(k+1)}| \approx |\Delta x^{(k)}|^{\rho}$ with $\rho > 1$; see [Iri81], for example. Durand-Kerner's method [Dur60, Ker66 and Abe73] calculates all the roots of $f(x)$ simultaneously by starting from initial approximations $x_i = x_i^{(0)}$ $(i = 1, \ldots, \deg_x(f))$. Kung-Traub's method for bivariate polynomial $F(x, y)$ corresponds to single-root Newton's method and, with an initial approximation $x = x^{(0)}$ where $x^{(0)}$ is a root of $F(x, 0)$, it calculates the higher power terms of power series iteratively. On the other hand, the method using parallel Hensel construction corresponds to Durand-Kerner's method.

In **2**, we review the generalized Hensel construction and its breakdown. In **3**, we consider calculating the power series roots of bivariate polynomial $F(x,y)$ in irregular case, by extending the generalized Hensel construction slightly. We will see that Puiseux series expansions of the roots are obtained by this extension. In **4**, we consider calculating the power series roots of multivariate polynomial $F(x,y,\ldots,z)$ in irregular case, by applying the extended Hensel construction. However, the application is not so direct: in the case of multivariate polynomials of more than two variables, different kinds of fractional-power series expansions are possible which show different analytic behaviors, and the expanded series are not so simple as those in bivariate case. We will investigate fractional-power series expansion of the roots, w.r.t. the total-degree of $y,\ldots,z$; for the total-degree, see **2**. Finally, in **5**, we discuss our method from the viewpoint of practical usefulness.

2 Hensel construction and its breakdown

Let K be a number field of characteristic 0. The field of complex numbers is denoted by $\mathbf{C}$. By $K[y,\ldots,z]$, $K(y,\ldots,z)$ and $K\{y,\ldots,z\}$, we denote the polynomial ring, rational function field and power series ring, respectively, over K in variables $y,\ldots,z$. By $(A_1,\ldots,A_m)$, with $A_1,\ldots,A_m$ polynomials or algebraic functions, we denote an ideal generated by $A_1,\ldots,A_m$.

Let $F(x,y,\ldots,z) \in K[x,y,\ldots,z]$. By $\deg_x(F)$ and $\deg_y(F)$, we denote the degrees of F with respect to x and y, respectively. Let $T = cx_1^{e_1}\cdots x_n^{e_n}$, with $c \in K$. By $\mathrm{tdeg}(T)$, we denote the total-degree of T, i.e., $\mathrm{tdeg}(T) = e_1 + \cdots + e_n$. Total-degree of a polynomial is the maximum of total-degrees of its terms. The greatest common divisor of polynomials F and G is denoted by $\gcd(F,G)$. Let $F(x,y,\ldots,z)$ be expressed as

$$F(x,y,\ldots,z) = f_d(y,\ldots,z)x^d + f_{d-1}(y,\ldots,z)x^{d-1} + \cdots + f_0(y,\ldots,z). \qquad (2.1)$$

If $f_d = 1$ then F is called *monic* w.r.t. x. If F has no multiple root w.r.t. x, then F is called *square-free* w.r.t. x. As is well known,

$$F \text{ is square-free w.r.t. } x \iff \gcd(F, \mathrm{d}F/\mathrm{d}x) = 1. \qquad (2.2)$$

In this paper, we calculate the roots by expanding them at $(y=0,\ldots,z=0)$. This restriction does not reduce the generality because the expansion at $(y=y_0,\ldots,z=z_0)$, with $y_0,\ldots,z_0$ nonzero numbers, is nothing but the expansion of the roots of $F'(x,y,\ldots,z) = F(x,y+y_0,\ldots,z+z_0)$ at $(y=0,\ldots,z=0)$. Furthermore, when $F(x,y,\ldots,z)$ is not monic w.r.t. x, we transform F into a monic polynomial $\tilde{F}$ by the following well-known transformation.

$$F(x,y,\ldots,z) \implies \tilde{F}(x,y,\ldots,z) = f_d^{d-1}F(x/f_d,y,\ldots,z). \qquad (2.3)$$

Then, the roots $\chi_i(y,\ldots,z)$ $(i=1,\ldots,d)$ of $F(x,y,\ldots,z)$ w.r.t. x can be expressed by the roots $\tilde{\chi}_i(y,\ldots,z)$ $(i=1,\ldots,d)$ of $\tilde{F}(x,y,\ldots,z)$ w.r.t. x as

$$\chi_i(y,\ldots,z) = \tilde{\chi}_i(y,\ldots,z)/f_d(y,\ldots,z), \quad i=1,\ldots,d. \qquad (2.4)$$

Therefore, the problem is reduced to finding the roots of monic polynomial. Note that, if $f_d(y,\ldots,z)$ contains a constant term, hence $f_d(0,\ldots,0) \neq 0$, then we can calculate f_d^{-1} in $K\{y,\ldots,z\}$ so we can convert F into monic by multiplying f_d^{-1} to F.

Next, let us consider the condition in (1.2), i.e., square-freeness of $F(x, 0, \dots, 0)$. As is well-known, decomposition of F into square-free factors, or *square-free decomposition* of F can be done by the gcd and exact division operations, using relation (2.2). Hence, we may assume without loss of generality that the given polynomial $F(x, y, \dots, z)$ is square-free. Note that, even with this assumption, $F(x, 0, \dots, 0)$ may not be square-free.

Let $F(x, y, \dots, z)$ be a monic square-free polynomial in $K[x, y, \dots, z]$, and $F(x, 0, \dots, 0)$ be factorized in $K[x]$ as

$$\begin{aligned} &F(x, 0, \dots, 0) = G_1^{(0)}(x) \cdots G_r^{(0)}(x), \quad r \geq 2, \\ &G_i^{(0)} \text{ and } G_j^{(0)} \text{ are relatively prime for any } \ i \neq j. \end{aligned} \tag{2.5}$$

The generalized Hensel construction is to construct $G_i^{(k)}(x, y, \dots, z)$ $(i = 1, \dots, r)$ in $K\{y, \dots, z\}[x]$, successively for $k = 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow \ \cdots$, satisfying

$$\begin{aligned} &F(x, y, \dots, z) \equiv G_1^{(k)}(x, y, \dots, z) \cdots G_r^{(k)}(x, y, \dots, z) \pmod{(y, \dots, z)^{k+1}}, \\ &\deg_x(G_i^{(k)}) = \deg_x(G_i^{(0)}), \quad G_i^{(k)}(x, 0, \dots, 0) = G_i^{(0)}(x), \quad i = 1, \dots, r. \end{aligned} \tag{2.6}$$

See [Hen08] for the Hensel construction and [Lau82 and GCL92] for the generalized Hensel construction. Note that the relative primality of $G_i^{(0)}(x)$ and $G_j^{(0)}(x)$ for any $i \neq j$, is crucial in the above method.

In the actual computation, we set $K = \mathbf{C}$, so $F(x, 0, \dots, 0)$ is factorized as

$$\begin{aligned} &F(x, 0, \dots, 0) = (x - \zeta_1)^{m_1} \cdots (x - \zeta_r)^{m_r}, \\ &\zeta_1, \dots, \zeta_r \in \mathbf{C}, \quad \zeta_i \neq \zeta_j \ \text{ for any } \ i \neq j. \end{aligned} \tag{2.7}$$

Then, we put $G_i^{(0)}(x) = (x - \zeta_i)^{m_i}$ $(i = 1, \dots, r)$, and the Hensel construction can be performed if $r \geq 2$. Since $\deg_x(G_i^{(k)}) = m_i$ and $G_i^{(k)}$ is monic w.r.t. x, we have $G_i^{(k)}(x, y, \dots, z) = x - \chi_i^{(k)}(y, \dots, z)$ if $m_i = 1$. That is, the Hensel construction gives the root corresponding to ζ_i if $m_i = 1$. However, if $m_i \geq 2$, we must factorize $G_i^{(k)}$ further into linear factors to get the roots, which cannot be done by the conventional Hensel construction. Therefore, only one case which we must consider and for which the conventional Hensel construction breaks down is that

$$F(x, 0, \dots, 0) = (x - \zeta_1)^{m_1} \cdots (x - \zeta_r)^{m_r}, \quad \max\{m_1, \dots, m_r\} \geq 2. \tag{2.8}$$

3 Calculating the roots of $F(x, y)$

First of all, we note that the Hensel construction is applicable to $F(x, y, \dots, z) \in K\{y, \dots, z\}[x]$. Hence, we assume in this section that $F(x, y) \in \mathbf{C}\{y\}[x]$, including the case of $F(x, y) \in \mathbf{C}[x, y]$. This is necessary because, in the process of root calculation, we first factorize $F(x, y, \dots, z)$ in $\mathbf{C}\{y, \dots, z\}[x]$ by applying Hensel construction, obtaining $G^{(k)}(x, y, \dots, z) \in \mathbf{C}\{y, \dots, z\}[x]$ as a factor of F (see **3.2**), then calculate the roots of $G^{(k)}(x, y, \dots, z)$. Of course, the $F(x, y, \dots, z)$ given initially is a polynomial.

We assumed that the polynomial given initially was monic w.r.t. x and square-free. Hence, we assume that $F(x, y) \in \mathbf{C}\{y\}[x]$ is also monic and has no multiple root w.r.t. x.

Furthermore, as the case for which the conventional Hensel construction breaks down, we may assume without loss of generality that

$$F(x,0) = x^d. \tag{3.1}$$

3.1 Initial factors and modulus

In order to perform the Hensel construction, we need relatively prime polynomials as initial factors. We determine the initial factors by adding some terms containing y to the leading term x^d. The additional terms are determined by the method of Newton polygon which we call Newton's line (Newton's line is a side of the Newton polygon).

Definition 1 (Newton's line) *For each nonzero term $cx^{e_x}y^{e_y}$ of $F(x,y)$, we plot a dot at the point (e_x, e_y) in the two-dimensional Cartesian coordinate system. Let L be a straight line such that it passes the point $(d,0)$ as well as another dot plotted and that no dot plotted is below L; see Fig. 1. The line L is called Newton's line for F.*

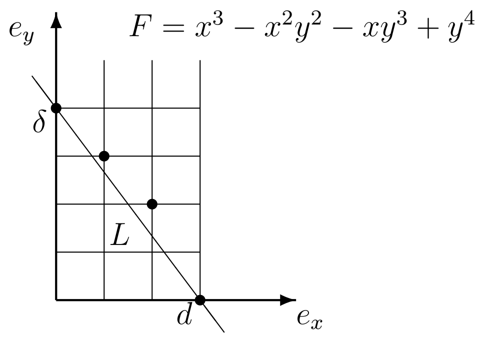

Fig 1. Illustration of Newton's Line L

Definition 2 (Newton's polynomial) *The sum of all the terms of $F(x,y)$, which are plotted on Newton's line is called Newton's polynomial for F.*

Note that Newton's line is uniquely determined by $F(x,y)$. Let δ be the e_y-coordinate of intersection of L and e_y-axis, hence Newton's line is $e_x/d + e_y/\delta = 1$. Let $F^{(0)}(x,y)$ be Newton's polynomial for $F(x,y)$, hence $F^{(0)}(x,y)$ consists of some of the terms

$$x^d,\ x^{d-1}y^{\delta/d},\ x^{d-2}y^{2\delta/d},\ \ldots,\ y^{d\delta/d}.$$

Let $F^{(0)}(x,1)$ be factorized over $\mathbf{C}$ as

$$\begin{aligned} &F^{(0)}(x,1) = (x-\zeta_1)^{m_1}\cdots(x-\zeta_r)^{m_r}, \\ &\zeta_1,\ldots,\zeta_r \in \mathbf{C}, \quad \zeta_i \neq \zeta_j \ \text{ for any } \ i \neq j. \end{aligned} \tag{3.2}$$

Then, since $F^{(0)}(x,y)$ is a homogeneous polynomial in x and $y^{\delta/d}$, $F^{(0)}(x,y)$ is factorized in $\mathbf{C}[x, y^{\delta/d}]$ as

$$F^{(0)}(x,y) = (x-\zeta_1 y^{\delta/d})^{m_1}\cdots(x-\zeta_r y^{\delta/d})^{m_r}. \tag{3.3}$$

If $r \geq 2$, we will use these factors as initial factors of Hensel construction.

Let $\hat{\delta}$ and $\hat{d}$ be positive integers such that

$$\hat{\delta}/\hat{d} = \delta/d, \quad \gcd(\hat{\delta}, \hat{d}) = 1. \tag{3.4}$$

Remark 1 We have $1 \leq \hat{d} \leq d$, because Newton's line passes a point (e_1, e_y), where e_x and e_y are integers satisfying $0 \leq e_x < d$, $0 < e_y$ and $\delta/d = e_y/(d - e_x)$.

Lemma 1 *Let $F(x, y)$ be an element of $\mathbf{C}\{y\}[x]$, and let $\hat{\delta}$ and $\hat{d}$ be defined as above by using Newton's line for $F(x, y)$. Then, except for numerical coefficients, any term of $F(x, y)$ is contained in the set*

$$\{x^d y^{(k+0)/\hat{d}},\ x^{d-1} y^{(k+\hat{\delta})/\hat{d}},\ x^{d-2} y^{(k+2\hat{\delta})/\hat{d}},\ \ldots,\ x^0 y^{(k+d\hat{\delta})/\hat{d}} \mid k = 0, 1, 2, \ldots\}.$$

Proof. Consider a term $x^{e_x} y^{e_y}$, where e_x and e_y are any integers such that $0 \leq e_x \leq d$, $0 \leq e_y$ and that (e_x, e_y) is not below Newton's line L. The latter condition implies $e_x/d + e_y/\delta \geq 1$. In order to prove the lemma, it is enough to show that there exist integers i and k satisfying conditions $0 \leq i \leq d$, $0 \leq k$, and $x^{d-i} y^{(k+i\hat{\delta})/\hat{d}} = x^{e_x} y^{e_y}$. The last condition is satisfied by putting $i = d - e_x$ and $k = e_y \hat{d} - i\hat{\delta}$, so we have $0 \leq i \leq d$. The condition $k \geq 0$ is also satisfied, because $e_x/d + e_y/\delta \geq 1$ means

$$\frac{k}{\hat{\delta} d} = \frac{e_y \hat{d} - (d - e_x)\hat{\delta}}{\hat{\delta} d} = \frac{e_x}{d} + \frac{e_y}{\delta} - 1 \geq 0.$$

This proves the lemma. $\square$

Lemma 1 leads us to define ideals S_k $(k{=}1, 2, \ldots)$ in $\mathbf{C}\{y^{1/\hat{d}}\}[x]$ as follows.

$$\begin{aligned} S_k &= (x, y^{\hat{\delta}/\hat{d}})^d \times (y^{1/\hat{d}})^k \\ &= (x^d,\ x^{d-1} y^{\hat{\delta}/\hat{d}},\ x^{d-2} y^{2\hat{\delta}/\hat{d}},\ \ldots,\ x^0 y^{d\hat{\delta}/\hat{d}}) \times (y^{k/\hat{d}}) \\ &= (x^d y^{(k+0)/\hat{d}},\ x^{d-1} y^{(k+\hat{\delta})/\hat{d}},\ x^{d-2} y^{(k+2\hat{\delta})/\hat{d}},\ \ldots,\ x^0 y^{(k+d\hat{\delta})/\hat{d}}). \end{aligned} \tag{3.5}$$

We use S_k $(k{=}1, 2, \ldots)$ as moduli of Hensel construction.

3.2 Extended Hensel construction

In this subsection, given $F(x, y) \in \mathbf{C}\{y\}[x]$, we consider factorizing $F(x, y)$ as $F(x, y) = G_1(x, y) \cdots G_r(x, y)$ in $\mathbf{C}\{y^{1/\hat{d}}\}[x]$, with $\deg_x(G_i) \geq 1$ $(i = 1, \ldots, r)$ in general. We assumed that $F^{(0)}(x, y)$ was factorized as in (3.3), so we put

$$\begin{aligned} &F^{(0)}(x, y) = G_1^{(0)}(x, y) \cdots G_r^{(0)}(x, y), \\ &G_i^{(0)}(x, y) = (x - \zeta_i y^{\hat{\delta}/\hat{d}})^{m_i}, \quad i = 1, \ldots, r. \end{aligned} \tag{3.6}$$

Furthermore, for simplicity, we put

$$\hat{y} = y^{\hat{\delta}/\hat{d}}. \tag{3.7}$$

Lemma 2 (Lagrange's interpolation polynomials) *Let* $\hat{G}_i(x,\hat{y})$ $(i=1,\dots,r)$ *be homogeneous polynomials in* x *and* $\hat{y}$*, with* $r \ge 2$ *and* $\deg_x(\hat{G}_i) = m_i$ $(i=1,\dots,r)$*, such that*

$$\gcd(\hat{G}_i, \hat{G}_j) = 1 \quad \text{for any} \quad i \neq j. \tag{3.8}$$

Then, for each $l \in \{0,\dots,d-1\}$*, where* $d = \deg_x(\hat{G}_1 \cdots \hat{G}_r)$*, there exists only one set of polynomials* $\{W_i^{(l)}(x,\hat{y}) \mid i=1,\dots,r\}$ *satisfying*

$$\begin{aligned} &W_1^{(l)}[\hat{G}_1 \cdots \hat{G}_r/\hat{G}_1] + \cdots + W_r^{(l)}[\hat{G}_1 \cdots \hat{G}_r/\hat{G}_r] = x^l \hat{y}^{d-l}, \\ &\deg_x(W_i^{(l)}(x,\hat{y})) < \deg_x(\hat{G}_i(x,\hat{y})), \quad i=1,\dots,r. \end{aligned} \tag{3.9}$$

$W_i^{(0)},\dots,W_i^{(d-1)}$ $(i=1,\dots,r)$ *are homogeneous polynomials in* x *and* $\hat{y}$*, of total-degree* m_i*. We call* $W_i^{(l)}$ $(i=1,\dots,r)$ *Lagrange's interpolation polynomials.*

Proof. There exists only one set of polynomials $\{W_i^{(l)} \mid i=1,\dots,r\}$ satisfying (3.9) with $\hat{y}=1$. (For the proof, see [Lau82] or [SS92].) Since $\deg_x(W_i^{(l)}[\hat{G}_1 \cdots \hat{G}_r/\hat{G}_i]) < d$, we can homogenize $W_i^{(l)}(x,1)$ and $\hat{G}_i(x,1)$ $(i=1,\dots,r)$ simultaneously, which gives $W_i^{(l)}(x,\hat{y})$ $(i=1,\dots,r)$ uniquely. Since the r.h.s. of the upper equality of (3.9) is of total-degree d w.r.t. x and $\hat{y}$, $W_i^{(l)}$ is of total-degree m_i w.r.t. x and $\hat{y}$. □

Theorem 1 *Let* $F(x,y)$ *be a monic square-free polynomial in* x *with coefficients in* $K\{y\}$*. Let* $F^{(0)}(x,y)$ *be Newton's polynomial for* $F(x,y)$ *and factorized as in* (3.6) *with* $r \ge 2$*, and let* S_k $(k=1,2,\dots)$ *be defined by* (3.5)*. Then, for any positive integer* k*, we can construct* $G_i^{(k)}(x,y) \in \mathbf{C}\{y^{1/\hat{d}}\}[x]$ $(i=1,\dots,r)$*, satisfying*

$$F(x,y) \equiv G_1^{(k)}(x,y) \cdots G_r^{(k)}(x,y) \pmod{S_{k+1}}, \tag{3.10}$$

$$G_i^{(k)}(x,y) \equiv G_i^{(0)}(x,y) \pmod{S_1}, \quad i=1,\dots,r. \tag{3.11}$$

Proof. By mathematical induction on k.
Since $F(x,y) \equiv F^{(0)}(x,y) \pmod{S_1}$, the theorem is valid for $k=0$. We note that $G_i^{(0)}(x,y)$ $(i=1,\dots,r)$ are monic w.r.t. x and homogeneous w.r.t. x and $\hat{y}$. Suppose that the theorem is valid up to the (k–1)-st construction step ($k \ge 1$). Furthermore, as induction assumptions, we assume that $G_i^{(k-1)}$ is expressed as

$$\begin{aligned} &G_i^{(k-1)}(x,y) = G_i^{(0)}(x,y) + \Delta G_i^{(1)}(x,y) + \cdots + \Delta G_i^{(k-1)}(x,y), \\ &\deg_x(\Delta G_i^{(k')}(x,y)) < \deg_x(G_i^{(0)}(x,y)) = m_i, \quad k'=1,\dots,k-1, \end{aligned} \tag{3.12}$$

and that $\Delta G_i^{(k')}(x,y)/y^{k'/\hat{d}}$ $(k'=1,\dots,k-1)$ are homogeneous polynomials in x and $\hat{y}$, of total-degree m_i w.r.t. x and $\hat{y}$. We put

$$\Delta F^{(k)}(x,y) \equiv F(x,y) - G_1^{(k-1)}(x,y) \cdots G_r^{(k-1)}(x,y) \pmod{S_{k+1}}. \tag{3.13}$$

Then, Lemma 1 and induction assumptions tell us that $\Delta F^{(k)}(x,y)$ is expressed as

$$\begin{aligned} &\Delta F^{(k)}(x,y) = f_{d-1}^{(k)}(y) \cdot x^{d-1} y^{\hat{\delta}/\hat{d}} + \cdots + f_0^{(k)}(y) \cdot x^0 y^{d\hat{\delta}/\hat{d}}, \\ &f_l^{(k)}(y) = c_l^{(k)} y^{k/\hat{d}}, \quad c_l^{(k)} \in \mathbf{C} \quad (l=0,\dots,d-1). \end{aligned} \tag{3.14}$$

We construct $G_i^{(k)}(x,y)$ $(i=1,\ldots,r)$ by putting

$$G_i^{(k)}(x,y) = G_i^{(k-1)}(x,y) + \Delta G_i^{(k)}(x,y), \quad \Delta G_i^{(k)} \equiv 0 \pmod{S_k}. \tag{3.15}$$

Substituting this expression for $G_i^{(k)}$ in (3.10), and noting that $\Delta G_i^{(k)} \equiv 0 \pmod{S_k}$, we see that (3.10) is satisfied if and only if the following equation is satisfied.

$$\Delta F^{(k)} \equiv \Delta G_1^{(k)}[G_2^{(0)} \cdots G_r^{(0)}] + \cdots + \Delta G_r^{(k)}[G_1^{(0)} \cdots G_{r-1}^{(0)}] \pmod{S_{k+1}}. \tag{3.16}$$

Lemma 2, with $\hat{G}_i(x,\hat{y}) = G_i^{(0)}(x,y)$ $(i=1,\ldots,r)$, and $\Delta F^{(k)}$ in (3.14) allow us to solve this equation as

$$\Delta G_i^{(k)}(x,y) = \sum_{l=0}^{d-1} W_i^{(l)}(x,y) f_l^{(k)}(y), \quad i = 1,\ldots,r, \tag{3.17}$$

where $W_i^{(l)}$ $(i=1,\ldots,r)$ are defined in Lemma 2. The $\Delta G_i^{(k)}$ thus calculated is of degree less than m_i w.r.t. x and $\Delta G_i^{(k)}(x,y)/y^{k/\hat{d}}$ is a homogeneous polynomial in x and $\hat{y}$, of total-degree m_i w.r.t. x and $\hat{y}$. This completes the proof. □

Corollary 1 $G_i^{(k)}$ $(i=1,\ldots,r)$ *satisfying* (3.10) *with* (3.11) *are unique.*

Proof. This follows directly from the uniqueness of $W_i^{(0)}(x,y),\ldots,W_i^{(d-1)}(x,y)$ $(i=1,\ldots,r)$. □

We call the above-mentioned construction *extended Hensel construction.*

In the rest of this subsection, we show that if $F^{(0)}(x,y)$ is factorized in $\mathbf{C}[x,y]$ then $F(x,y)$ is factorized in $\mathbf{C}\{y\}[x]$. In order to show this, we have only to show that extended Hensel construction can be performed in $\mathbf{C}[x,y]$ in this case.

Theorem 2 *Let* $F^{(0)}(x,y)$ *be factorized in* $\mathbf{C}[x,y]$ *as*

$$\begin{aligned} &F^{(0)}(x,y) = F_1^{(0)}(x,y) \cdots F_s^{(0)}(x,y), \quad s \geq 2, \\ &\gcd(F_i^{(0)}, F_j^{(0)}) = 1 \quad \textit{for any} \;\; i \neq j. \end{aligned} \tag{3.18}$$

Let the extended Hensel construction of $F(x,y)$*, with initial factors* $F_1^{(0)}(x,y),\cdots,F_s^{(0)}(x,y)$*, be as follows* ($k$ *is any positive integer*).

$$F(x,y) \equiv F_1^{(k)}(x,y) \cdots F_s^{(k)}(x,y) \pmod{S_{k+1}}. \tag{3.19}$$

Then, $F_i^{(k)}(x,y) \in \mathbf{C}[x,y]$ $(i=1,\ldots,s)$.

Proof. Lagrange's interpolation polynomials $W_i^{(0)},\ldots,W_i^{(d-1)}$ $(i=1,\ldots,s)$, satisfying (3.9) with $\hat{G}_i = F_i^{(0)}$, are calculated by the extended Euclidean algorithm which consists of only rational operations. Hence, $W_i^{(l)} \in K(y)[x]$ $(l=0,\ldots,d-1)$. On the other hand, Lemma 2 says that $W_i^{(l)}$ is homogeneous w.r.t. x and $\hat{y}$. Hence, $W_i^{(l)} \in K[x,y]$. Now, the theorem is apparently valid for $k=0$. Suppose that it is valid up to the $(k-1)$-st construction step $(k \geq 1)$, then putting $\Delta F^{(k)} \equiv F - F_1^{(k-1)} \cdots F_s^{(k-1)} \pmod{S_{k+1}}$, we find that $\Delta F^{(k)} \in \mathbf{C}[x,y]$. Hence, the construction formula (3.17) tells us that $F_i^{(k)}$ ($i = 1,\ldots,s$) are also in $\mathbf{C}[x,y]$. $\square$

Remark 2 If the slope of Newton's line is an integer then $\hat{d} = 1$ and $\hat{y} = y^{\hat{\delta}}$, so $F^{(0)}(x,y)$ is obviously factorized in $\mathbf{C}\{y\}[x]$. However, Theorem 2 is valid regardless of whether $\hat{d} = 1$ or not (see Example 1, below).

In the actual root calculation, $F^{(0)}(x,y)$ should first be factorized as in (3.18) then be factorized as in (3.6), although we have stated Theorem 1 earlier than Theorem 2.

3.3 Example of extended Hensel construction

Example 1 $F(x,y) = x^5 + x^4y - 2x^3y - 2x^2y^2 + x(y^2 - y^3) + y^3$.

We have $d = \deg_x(F(x,y)) = 5$. Newton's line L is determined to be $e_x/5 + e_y/2.5 = 1$ with $\delta = 2.5$, hence $\delta/d = 1/2 = \hat{\delta}/\hat{d}$ and $\hat{\delta} = 1$ and $\hat{d} = 2$. The ideal S_0 is

$$S_0 = (x, y^{1/2})^5 = (x^5,\ x^4y^{1/2},\ x^3y,\ x^2y^{3/2},\ xy^2, y^{5/2}).$$

Newton's polynomial $F^{(0)}(x,y)$ is determined and factorized as

$$F^{(0)}(x,y) = x^5 - 2x^3y + xy^2 = x(x + y^{1/2})^2(x - y^{1/2})^2.$$

Hence, we put $G_1^{(0)} = x$, $G_2^{(0)} = (x + y^{1/2})^2$, $G_3^{(0)} = (x - y^{1/2})^2$.

Lagrange's interpolation polynomials are calculated as

$$\begin{array}{llll}
W_1^{(0)} = y^{1/2}\ , & W_2^{(0)} = -\frac{1}{2}xy^{1/2} - \frac{3}{4}y\ , & W_3^{(0)} = -\frac{1}{2}xy^{1/2} + \frac{3}{4}y\ , \\
W_1^{(1)} = 0\ , & W_2^{(1)} = \frac{1}{4}xy^{1/2} + \frac{1}{2}y\ , & W_3^{(1)} = -\frac{1}{4}xy^{1/2} + \frac{1}{2}y\ , \\
W_1^{(2)} = 0\ , & W_2^{(2)} = -\frac{1}{4}y\ , & W_3^{(2)} = \frac{1}{4}y\ , \\
W_1^{(3)} = 0\ , & W_2^{(3)} = -\frac{1}{4}xy^{1/2}\ , & W_3^{(3)} = \frac{1}{4}xy^{1/2}\ , \\
W_1^{(4)} = 0\ , & W_2^{(4)} = \frac{1}{2}xy^{1/2} + \frac{1}{4}y\ , & W_3^{(4)} = \frac{1}{2}xy^{1/2} - \frac{1}{4}y\ .
\end{array}$$

One can check easily that these polynomials satisfy (3.9).

For $S_2 = (x^5y,\ x^4y^{3/2},\ x^3y^2, x^2y^{5/2},\ xy^3,\ y^{7/2})$, we have

$$\begin{aligned}
\Delta F^{(1)} &\equiv F - G_1^{(0)}G_2^{(0)}G_3^{(0)} \pmod{S_2} \\
&= y^{1/2}\cdot x^4y^{1/2} - 2y^{1/2}\cdot x^2y^{3/2} + y^{1/2}\cdot y^{5/2}.
\end{aligned}$$

Hence, $f_4^{(1)} = y^{1/2}$, $f_2^{(1)} = -2y^{1/2}$, $f_0^{(1)} = y^{1/2}$ and $f_3^{(1)} = f_1^{(1)} = 0$. Using these polynomials (in $\hat{y} = y^{1/2}$) and formula in (3.17), we obtain

$$\begin{aligned}
G_1^{(1)} &= G_1^{(0)} + W_1^{(0)}f_0^{(1)} &&= x + y, \\
G_2^{(1)} &= G_2^{(0)} + W_2^{(4)}f_4^{(1)} + W_2^{(2)}f_2^{(1)} + W_2^{(0)}f_0^{(1)} &&= (x + y^{1/2})^2, \\
G_3^{(1)} &= G_3^{(0)} + W_3^{(4)}f_4^{(1)} + W_3^{(2)}f_2^{(1)} + W_3^{(0)}f_0^{(1)} &&= (x - y^{1/2})^2.
\end{aligned}$$

For $S_3 = (x^5y^{3/2},\ x^4y^2,\ x^3y^{5/2},\ x^2y^3,\ xy^{7/2},\ y^4)$, we have

$$\begin{aligned} \mathit{\Delta} F^{(2)} &\equiv F - G_1^{(1)}G_2^{(1)}G_3^{(1)} \pmod{S_3} \\ &= -y \cdot xy^2. \end{aligned}$$

Hence, $f_1^{(2)} = -y$ and $f_4^{(2)} = f_3^{(2)} = f_2^{(2)} = f_0^{(2)} = 0$ and we obtain

$$\begin{aligned} G_1^{(2)} &= G_1^{(1)} + 0 &&= x + y, \\ G_2^{(2)} &= G_2^{(1)} + W_2^{(1)} f_1^{(2)} &&= (x + y^{1/2})^2 - (\frac{1}{4}xy^{3/2} + \frac{1}{2}y^2), \\ G_3^{(2)} &= G_3^{(1)} + W_3^{(1)} f_1^{(2)} &&= (x - y^{1/2})^2 + (\frac{1}{4}xy^{3/2} - \frac{1}{2}y^2). \end{aligned}$$

Continuing two more iterations, we obtain

$$\begin{aligned} G_1^{(4)} &= x + y + y^2, \\ G_2^{(4)} &= (x + y^{1/2})^2 - (\frac{1}{4}xy^{3/2} + \frac{1}{2}y^2) - (\frac{1}{2}xy^2 + \frac{3}{4}y^{5/2}) - (\frac{53}{64}xy^{5/2} + \frac{9}{8}y^3), \\ G_3^{(4)} &= (x - y^{1/2})^2 + (\frac{1}{4}xy^{3/2} - \frac{1}{2}y^2) - (\frac{1}{2}xy^2 - \frac{3}{4}y^{5/2}) + (\frac{53}{64}xy^{5/2} - \frac{9}{8}y^3). \end{aligned}$$

We note that $G_2^{(4)}$ and $G_3^{(4)}$ can be written as

$$\begin{aligned} G_2^{(4)} &= G_P^{(4)} + y^{1/2}G_A^{(4)}, \\ G_3^{(4)} &= G_P^{(4)} - y^{1/2}G_A^{(4)}, \end{aligned}$$

where $G_P^{(4)}$ and $G_A^{(4)}$ are given by

$$\begin{aligned} G_P^{(4)} &= (x^2 + y) - \frac{1}{2}y^2 - \frac{1}{2}xy^2 - \frac{9}{8}y^3, \\ G_A^{(4)} &= 2x - \frac{1}{4}xy - \frac{3}{4}y^2 - \frac{53}{64}xy^2. \end{aligned}$$

The above computation suggests us that $G_1^{(\infty)} \in \mathbf{C}[x, y]$. In fact, this suggestion is true by Theorem 2 because $F^{(0)}(x, y)$ in the above example is factorized in $\mathbf{C}[x, y]$ as $F^{(0)}(x, y) = x\,(x^4 - 2x^2y + y^2)$. Furthermore, Theorem 5 to be presented in the next section will show that $G_P^{(\infty)}, G_A^{(\infty)} \in \mathbf{C}[x, y]$.

3.4 Calculating the roots in $\mathbf{C}\{y^{1/\bar{d}}\}$

In this subsection, we consider factorizing $F(x, y)$ into linear factors over $\mathbf{C}$ as

$$\begin{aligned} &F(x, y) = \bar{F}_1(x, y) \cdots \bar{F}_d(x, y) = (x - \chi_1(y)) \cdots (x - \chi_d(y)), \\ &\chi_i(y) \in \mathbf{C}\{y^{1/\bar{d}_i}\} \ \text{ with } \bar{d}_i \text{ a positive integer}, \quad i = 1, \ldots, d, \end{aligned} \tag{3.20}$$

where $F(x,y)$ is in $\mathbf{C}\{y\}[x]$, monic w.r.t. x and square-free. We first simplify the problem using Theorem 2.

As the factorization in (3.18), suppose that we have the following.

$$F^{(0)}(x,y) = H_1(x,y)^{m_1} H_2(x,y)^{m_2} \cdots H_s(x,y)^{m_s} \qquad (3.21)$$
$$\text{(irreducible factorization in } \mathbf{C}[x,y]),$$

$$\gcd(H_i, H_j) = 1 \quad \text{for any} \quad i \neq j. \qquad (3.22)$$

The degree of H_i is related with $\hat{d}$, as the following lemma shows.

Lemma 3 *Let $F^{(0)}(x,y)$ be factorized as in* (3.21) *with* (3.22)*. Then,*

$$\hat{d} \mid \deg_x(H_i) \quad \textit{for any} \quad i \quad \textit{such that} \quad H_i \neq x. \qquad (3.23)$$

Proof. Since $H_i(x,y)$ is irreducible in $\mathbf{C}[x,y]$, $H_i(x,y)$ must be of the form

$$H_i(x,y) = x^{r_i} + \cdots + c_0 y^{\rho_i}, \quad c_0 \neq 0, \quad \rho_i/r_i = \delta/d.$$

Here, r_i and ρ_i are nonzero integers and $\hat{d}$ and $\hat{\delta}$ are the smallest positive integers satisfying $\hat{\delta}/\hat{d} = \delta/d$, hence we obtain (3.23). $\square$

Suppose that $s \geq 2$ in (3.21). Then, Theorem 2 assures that we can perform the factorization in $\mathbf{C}\{y\}[x]$, of $F(x,y)$ as follows.

$$\begin{aligned} &F(x,y) \equiv F_1^{(k)}(x,y) \cdots F_s^{(k)}(x,y) \pmod{S_{k+1}}, \\ &F_i^{(k)}(x,y) \equiv H_i(x,y)^{m_i} \pmod{S_1}, \quad i = 1, \ldots, s. \end{aligned} \qquad (3.24)$$

Remember that $H_i(x,y)$ is irreducible in $\mathbf{C}[x,y]$ and homogeneous w.r.t. x and $y^{\hat{\delta}/\hat{d}}$. Hence, for $H_i(x,y) \neq x$, we can factorize $H_i(x,1)$ as

$$H_i(x,1) = (x - \zeta_{i,1}) \cdots (x - \zeta_{i,r_i}), \quad \zeta_{i,j_1} \neq \zeta_{i,j_2} \ \text{ if } \ j_1 \neq j_2,$$

where $r_i = \deg_x(H_i)$. Therefore, we have

$$\begin{aligned} F_i(x,y) &\equiv H_i(x,y)^{m_i} \pmod{S_1} \\ &\equiv (x - \zeta_{i,1} y^{\hat{\delta}/\hat{d}})^{m_i} \cdots (x - \zeta_{i,r_i} y^{\hat{\delta}/\hat{d}})^{m_i} \pmod{S_1}. \end{aligned} \qquad (3.25)$$

Using the above $(x - \zeta_{i,j} y^{\hat{\delta}/\hat{d}})^{m_i}$ $(j = 1, \ldots, r_i)$ as initial factors of Hensel construction, we can factorize $F_i^{(k)}(x,y)$ in $\mathbf{C}\{y^{1/\hat{d}}\}[x]$ as described in **3.2**. Hence, in order to perform the factorization of $G(x,y)$ into linear factors, we have only to consider $G(x,y)$ such that

$$\begin{aligned} &G(x,y) \in \mathbf{C}\{y^{1/\hat{d}}\}[x], \quad G(x,y) \text{ is monic w.r.t. } x, \\ &G(x,y) \equiv (x - \zeta y^{\hat{\delta}/\hat{d}})^m \pmod{S_1}, \quad m \geq 2. \end{aligned} \qquad (3.26)$$

Of course, $G(x,y)$ corresponding to $H_i(x) = x$ is in $\mathbf{C}\{y\}[x]$.

Now, we describe a method of factorizing $G(x,y)$ into linear factors. First, we perform the following transformation for $G(x,y)$.

$$G(x,y) \Longrightarrow G'(x,y) = G(x+\zeta y^{\hat{\delta}/\hat{d}}, y). \tag{3.27}$$

Then, we have

$$G'(x,y) \in \mathbf{C}\{y^{1/\hat{d}}\}[x], \quad G'(x,y) \equiv x^m \pmod{S_1}. \tag{3.28}$$

Therefore, without loss of generality, we may assume that $\zeta = 0$ in (3.26), i.e.,

$$G(x,y) \equiv x^m \pmod{S_1}. \tag{3.29}$$

Eq. (3.29) is similar to (3.1), and we perform the factorization of $G(x,y)$ similarly to that described in **3.1** and **3.2**. That is, we draw Newton's line for $G(x,y)$, construct Newton's polynomial $G^{(0)}(x,y) \in \mathbf{C}[x, y^{1/\hat{d}}]$, factorize $G^{(0)}(x,y)$ over $\mathbf{C}$, and perform the extended Hensel construction. Only one thing that needs explanation is the determination of modulus of Hensel construction.

Let Newton's line, in the e_x-e_y coordinate plane, for $G(x,y)$ be $e_x/m + e_y/\mu = 1$, and let $\hat{m}$ and $\hat{\mu}$ be positive integers satisfying

$$\hat{\mu}/\hat{m} = \mu/m, \quad \gcd(\hat{\mu}, \hat{m}) = 1. \tag{3.30}$$

Then, instead of Lemma 1, we have the following lemma for $G(x,y)$.

Lemma 4 *Except for numerical coefficients, any term of $G(x,y)$ is contained in the set*

$$\{x^m y^{(k+0)/(\hat{d}\hat{m})},\ x^{m-1} y^{(k+\hat{d}\hat{\mu})/(\hat{d}\hat{m})},\ \ldots,\ x^0 y^{(k+m\hat{d}\hat{\mu})/(\hat{d}\hat{m})} \;\big|\; k = 0,1,2,\ldots\}. \tag{3.31}$$

Proof. A slight modification of the proof for Lemma 1 proves this. □

Lemma 4 leads us to define ideals $\bar{S}_k$ $(k=1,2,\ldots)$ in $\mathbf{C}\{y^{1/(\hat{d}\hat{m})}\}[x]$ as follows which we use as moduli of extended Hensel construction for $G(x,y)$.

$$\begin{aligned} \bar{S}_k &= (x, y^{\hat{\mu}/\hat{m}})^m \times (y^{1/(\hat{d}\hat{m})})^k \\ &= (x^m,\ x^{m-1} y^{\hat{\mu}/\hat{m}},\ x^{m-2} y^{2\hat{\mu}/\hat{m}},\ \ldots,\ x^0 y^{\mu}) \times (y^{k/(\hat{d}\hat{m})}). \end{aligned} \tag{3.32}$$

Remark 3 Since $\hat{m}$ is a divisor of m, we may think that each term in (3.31) is contained in $\mathbf{C}\{y^{1/(\hat{d}m)}\}[x]$.

Remark 4 Consider (3.21) and suppose that $H_1(x,y) = x$ so $H_i(x,y) \neq x$ for $i \geq 2$. Then, $d = m_1 + r_2 m_2 + \cdots + r_s m_s$ and Lemma 3 tells us that $\hat{d} \mid \gcd(r_2,\ldots,r_s)$. If $H_i(x,y) \neq x$ for every i, then we have $d = r_1 m_1 + \cdots + r_s m_s$ and $\hat{d} \mid \gcd(r_1,\ldots,r_s)$. By this, we can easily get an upper bound of the denominator of the fractional powers.

Now, we can state our main theorem.

Theorem 3 *Let $F(x,y) \in K[x,y]$ be monic w.r.t. x and square-free. Then, by repeated use of the extended Hensel construction, we can factorize $F(x,y)$ into the form (3.20). Furthermore, let Newton's polynomial $F^{(0)}(x,y)$ for $F(x,y)$ be factorized as in (3.21) with (3.22), and let $F(x,y)$ be factorized as in (3.24), where we put $H_1(x,y) = x$. (If $F^{(0)}(x,y)$ does not contain x as a factor then we ignore $F_1(x,y)$ below.) Then, we have*

$$F_1(x,y) = (x - \chi_1^{(1)}(y)) \cdots (x - \chi_{m_1}^{(1)}(y)), \tag{3.33}$$

$$\chi_j^{(1)}(y) \in \mathbf{C}\{y^{1/m_1}\} \quad (j = 1, \ldots, m_1),$$

$$F_i(x,y) = (x - \chi_1^{(i)}(y)) \cdots (x - \chi_{m_i r_i}^{(i)}(y)), \tag{3.34}$$

$$\chi_j^{(i)}(y) \in \mathbf{C}\{y^{1/m_i \hat{d}}\} \quad (j = 1, \ldots, m_i r_i), \quad 2 \le i \le s.$$

Proof. Most part of this theorem has already been proved, and we have only to show that we can factorize $F(x,y)$ into linear factors w.r.t. x. Applying the extended Hensel construction to $G(x,y)$ in (3.26), we can factorize it as $G(x,y) = \bar{G}_1(x,y) \cdots \bar{G}_\rho(x,y)$, $\bar{G}_i \in \mathbf{C}\{y^{1/(\hat{d}\hat{m})}\}[x]$ $(i{=}1,\ldots,\rho)$. The $\bar{G}_i$ may still be of degree greater than 1. Then, we apply the procedure described in this subsection to $\bar{G}_i$ again. This application is possible because, by the assumption of square-freeness of $F(x,y)$, every root of $F(x,y)$ is different from each other and we can construct Newton's polynomial for any $\bar{G}_i$ of degree ≥ 2. Therefore, we will finally obtain linear factors. $\square$

Example 2 Let $G(x,y)$ be the third factor $G_3(x,y)$ in Example 1.

$$G(x,y) = (x - y^{1/2})^2 + (\frac{1}{4}xy^{3/2} - \frac{1}{2}y^2) - (\frac{1}{2}xy^2 - \frac{3}{4}y^{5/2}) + (\frac{53}{64}xy^{5/2} - \frac{9}{8}y^3) - \cdots.$$

With the transformation $x \Rightarrow x + y^{1/2}$, $G(x,y)$ becomes

$$G(x,y) \Longrightarrow x^2 + (\frac{1}{4}xy^{3/2} - \frac{1}{4}y^2) - (\frac{1}{2}xy^2 - \frac{1}{4}y^{5/2}) + (\frac{53}{64}xy^{5/2} - \frac{19}{64}y^3) - \cdots.$$

We have $m = \deg_x(G) = 2$. Newton's line is $e_x/2 + e_y/2 = 1$, hence $\hat{m} = \hat{\mu} = 1$ and the extended Hensel construction can be performed in $\mathbf{C}\{y^{1/2}\}[x]$. Newton's polynomial is

$$G^{(0)}(x,y) = x^2 - y^2/4 = (x + y/2)(x - y/2),$$

so we define ideals $\bar{S}_k$ $(k{=}1,2,\ldots)$ as follows.

$$\bar{S}_k = (x,y)^2 \times (y^{1/2})^k = (x^2 y^{k/2},\ xy^{1+k/2},\ y^{2+k/2}).$$

Put $\bar{G}_1^{(0)} = x + y/2$ and $\bar{G}_2^{(0)} = x - y/2$. Lagrange's interpolation polynomials for $\{\bar{G}_1^{(0)}, \bar{G}_2^{(0)}\}$ are determined as

$$\begin{aligned} W_1^{(0)} &= -y\ , & W_2^{(0)} &= y\ , \\ W_1^{(1)} &= y/2\ , & W_2^{(1)} &= y/2\ . \end{aligned}$$

For $\bar{S}_2 = (x^2 y, xy^2, y^3)$, we have

$$\Delta\bar{G}^{(1)} \equiv G - \bar{G}_1^{(0)}\bar{G}_2^{(0)} \equiv (\frac{1}{4}y^{1/2}) \cdot xy + (\frac{1}{4}y^{1/2}) \cdot y^2 \pmod{\bar{S}_2}.$$

Hence, we can calculate $\bar{G}_i^{(1)} = \bar{G}_i^{(0)} + \Delta\bar{G}_i^{(1)}$ $(i=0,1)$ as follows.

$$\begin{aligned}
\bar{G}_1^{(1)} &= \bar{G}_1^{(0)} + W_1^{(1)} \cdot (\frac{1}{4}y^{1/2}) + W_1^{(0)} \cdot (\frac{1}{4}y^{1/2}) &= x + \frac{1}{2}y - \frac{1}{8}y^{3/2}, \\
\bar{G}_2^{(1)} &= \bar{G}_2^{(0)} + W_2^{(1)} \cdot (\frac{1}{4}y^{1/2}) + W_2^{(0)} \cdot (\frac{1}{4}y^{1/2}) &= x - \frac{1}{2}y + \frac{3}{8}y^{3/2}.
\end{aligned}$$

Similarly, for $\bar{S}_3 = (x^2y^{3/2}, xy^{5/2}, y^{7/2})$, we obtain

$$\begin{aligned}
\bar{G}_1^{(2)}(x,y) &= x + \frac{1}{2}y - \frac{1}{8}y^{3/2} + 0, \\
\bar{G}_2^{(2)}(x,y) &= x - \frac{1}{2}y + \frac{3}{8}y^{3/2} - \frac{1}{2}y^2,
\end{aligned}$$

satisfying $G(x,y) \equiv \bar{G}_1^{(2)}(x,y)\bar{G}_2^{(2)}(x,y) \pmod{\bar{S}_3}$. $\square$

Putting $\bar{G}_i^{(k)}(x,y) = x - \chi_i^{(k)}(y),\ i = 1,2$, we see that $\chi_i^{(k)}$ satisfies $F(\chi_i^{(k)}(y), y) \equiv 0 \pmod{y^{2+(k+1)/2}}$. Since the extended Hensel construction is unique, as Corollary 1 says, the series $\chi_i^{(k)}(y)$ must be Puiseux series of a root of $F(x,y)$.

4 Calculating the roots of $F(x, y, \ldots, z)$

Now, let us consider the multivariate case; here, by multivariate case, we mean that the given polynomial $F(x, y, \ldots, z)$ contains three or more variables. As in **3**, we assume that $F(x, y, \ldots, z)$ is monic w.r.t. x, square-free, and satisfies

$$F(x, 0, \ldots, 0) = x^d. \tag{4.1}$$

4.1 Various fractional-power series expansions

We note that in the multivariate case there are many different kinds of fractional-power series expansions which show different analytic behaviors. In this subsection, we show this fact explicitly by simple examples, which will help the reader to understand the fractional-power series expansion of the roots of multivariate polynomials, and explain why we investigate the power series expansion w.r.t. the total-degree.

In many cases, some roots can be expanded into power series of fractional powers for only one variable and of integral powers for other variables. Let a given polynomial be $F(x, y, z_1, \ldots, z_n)$, and assume that $F(x, y, 0, \ldots, 0)$ is factorized as follows.

$$\begin{aligned}
&F(x, y, 0, \ldots, 0) = F_1^{(0)}(x,y) \cdots F_r^{(0)}(x,y), \\
&F_i^{(0)}(x,y) = (x - \xi_i(y))^{m_i}, \quad i = 1, \ldots, r, \\
&\xi_i(y) \neq \xi_j(y) \quad \text{for any } i \neq j,
\end{aligned} \tag{4.2}$$

where $\xi_i(y) \in \mathbf{C}\{y^{1/\hat{d}}\}$, with $\hat{d}$ a positive integer. The above factorization, to any desired power of y, can be done by the method described in **3**. The roots $\xi_i(y)$ $(i=1, \ldots, d)$ are algebraic functions over $\mathbf{C}(y)$, and we have an algebraic extension field $\mathbf{C}(y, \xi_i)$. Therefore, using the extended Euclidean algorithm over $\mathbf{C}(y, \xi_1, \ldots, \xi_r)$, we can calculate Lagrange's interpolation polynomials $W_1^{(l)}(x, y), \ldots, W_r^{(l)}(x, y)$ $(l{=}0, \ldots, d{-}1)$ satisfying

$$\begin{aligned} W_1^{(l)}[F_1^{(0)} \cdots F_r^{(0)}/F_1^{(0)}] + \cdots + W_r^{(l)}[F_1^{(0)} \cdots F_r^{(0)}/F_r^{(0)}] &= x^l, \\ \deg_x(W_i^{(l)}(x, y)) < \deg_x(F_i^{(0)}(x, y)), \quad i = 1, \ldots, r. & \end{aligned} \tag{4.3}$$

Using $F_i^{(0)}(x, y)$ and $W_i^{(l)}(x, y)$, we can perform the *conventional* Hensel construction for $F(x, y, z_1, \ldots, z_n)$ with moduli $S_k = (z_1, \ldots, z_n)^k$ $(k = 1, 2, \ldots)$. The result is an integral-power series in $z_1, \ldots, z_n$. If $m_i = 1$, which occurs frequently, we obtain a desired expansion of the root corresponding to ξ_i. In this expansion, an expansion coefficient may become infinite as $y \to 0$. Let us convince ourselves of this by an example.

Example 3 Expansion using bivariate linear factors.

$$F(x, y, z) = x^2 + (y - z)x - y - z + y^2 - z^2. \tag{4.4}$$

Applying the extended Hensel construction, we can factorize $F(x, y, 0)$ as

$$\begin{aligned} &F(x, y, 0) = x^2 + yx - y + y^2 = (x - \xi_1(y))(x - \xi_2(y)), \\ &\xi_i(y) = -\frac{y}{2} - (-1)^i \sqrt{y}(1 - \frac{3}{8}y - \frac{9}{128}y^2 - \frac{27}{1024}y^3 - \cdots), \quad i = 1, 2. \end{aligned} \tag{4.5}$$

We put $F_i^{(0)}(x, y) = x - \xi_i(y)$ $(i{=}1, 2)$. Note that we have

$$\begin{aligned} &\xi_1 + \xi_2 = -y, \quad \xi_1\xi_2 = -y + y^2, \\ &\xi_i^2 = -y\xi_i + y - y^2 \quad (i = 1, 2). \end{aligned}$$

Using these relations only, we can perform the following calculations. Lagrange's interpolation polynomials $W_1^{(l)}, W_2^{(l)}$ $(l{=}0, 1)$ satisfying (4.3) are

$$\begin{aligned} W_1^{(0)} &= \frac{2\xi_1 + y}{4y - 3y^2}, \qquad W_1^{(1)} = \frac{-y\xi_1 + 2y - 2y^2}{4y - 3y^2}, \\ W_2^{(0)} &= \frac{2\xi_2 + y}{4y - 3y^2}, \qquad W_2^{(1)} = \frac{-y\xi_2 + 2y - 2y^2}{4y - 3y^2}. \end{aligned}$$

The Hensel construction of $F(x, y, z)$ with moduli $S_k = (z^k)$ $(k{=}1, 2, \cdots)$ gives us

$$\begin{aligned} F_1^{(k)}(x, y, z) &= x - \xi_1 + \frac{-2\xi_1 + (\xi_1 - 3)y + 2y^2}{4y - 3y^2}z + \cdots, \\ F_2^{(k)}(x, y, z) &= x - \xi_2 + \frac{-2\xi_2 + (\xi_2 - 3)y + 2y^2}{4y - 3y^2}z + \cdots. \end{aligned} \tag{4.6}$$

Thus, the roots of $F(x, y, z)$ are expanded into integral-power series in z. Similarly, we can express the roots in terms of integral-power series in y. □

$F_i^{(k)}(x, y, z)$ $(i{=}1, 2)$ in (4.6) show that

$$(\text{coefficient of } z \text{ in } F_i^{(k)}(x, y, z)) \longrightarrow \infty \quad \text{as } y \to 0.$$

On the other hand, at $y = 0$, $F(x, 0, z)$ is factorized as

$$
\begin{aligned}
&F(x, 0, z) = (x - \eta_1(z))(x - \eta_2(z)), \\
&\eta_i(z) = \frac{z}{2} - (-1)^i \sqrt{z}\Big(1 + \frac{5}{8}z - \frac{25}{128}z^2 + \frac{125}{1024}z^3 - \cdots\Big), \quad i = 1, 2.
\end{aligned}
\tag{4.7}
$$

We see that $F_i^{(k)}(x, y, z)$ in (4.6) does not converge to $(x - \eta_i(z))$ if we set $y \to 0$. This means that Hensel constructions with initial factors in (4.5) and (4.7) lead to series which show different analytic behaviors at $y = 0$ and $z = 0$. Note that the series in (4.6) are Taylor series w.r.t. z while the series in (4.7) are Puiseux series w.r.t. z.

If we expand the roots w.r.t. the total-degree, we obtain an expression which converges to (4.5) and (4.7), respectively, as $z \to 0$ and $y \to 0$.

Example 4 Expansion w.r.t. the total-degree.
We use the same polynomial as in Example 3, and introduce the total-degree variable t by the transformation $y \to ty$, $z \to tz$:

$$F(x, y, z) = x^2 + t(y - z)x - t(y + z) + t^2(y^2 - z^2).$$

The roots $\chi_i(y, z)$ of $F(x, y, z)$, w.r.t. x, are

$$\chi_i(y, z) \;=\; \frac{-t(y - z) - (-1)^i \sqrt{t^2(y - z)^2 + 4t(y + z) - 4t^2(y^2 - z^2)}}{2}, \quad i = 1, 2.$$

Expanding the roots into fractional-power series in t, we obtain

$$
\begin{aligned}
\chi_i(y, z) &= \frac{1}{2}t(z - y) - (-1)^i \sqrt{t(y + z)}\sqrt{1 - t(3y^2 + 2yz - 5z^2)/4(y + z)} \\
&= \frac{1}{2}t(z - y) - (-1)^i t^{1/2}\sqrt{y + z}\Big\{1 - \frac{1}{8}tR(y, z) - \frac{1}{128}t^2 R(y, z)^2 - \cdots\Big\},
\end{aligned}
$$

where $R(y, z) = (3y^2 + 2yz - 5z^2)/(y + z)$. Putting $t = 1$ in the above expressions, we obtain the required expansion. $\square$

We note that, in the above expressions, expansion coefficients do not become infinite if we set $y \to 0$ and/or $z \to 0$. Furthermore, we have the following property.

$$
\begin{aligned}
[\text{expansion of } F(x, y, z)] &\xrightarrow{y \to 0} [\text{expansion of } F(x, 0, z)] \\
&\xrightarrow{z \to 0} [\text{expansion of } F(x, y, 0)]
\end{aligned}
\tag{4.8}
$$

This convergence property seems to be desirable, however, the extended Hensel construction w.r.t. the total-degree does not always give this property. For example, consider

$$F(x, y, z) = x^2 + (y - z^3)x - (y + z^3) + (y^2 - z^6).$$

This polynomial is nothing but the one given in Example 4, with the replacement $z \to z^3$, and both $F(x, y, 0)$ and $F(x, 0, z)$ are expanded into fractional-power series. However, expansion w.r.t. the total-degree will lead to an expression in $\mathbf{C}\{y^{1/2}, z\}[x]$. If we define the total-degree with weights 1 and 1/3 for y and z, respectively, then we obtain the expansion satisfying (4.8).

It should be noted that the property (4.8) cannot always be attained even if we define the total-degree with any weights for y and z, as the following example shows.

Example 5 Expansion w.r.t. total-degree of various weights.

$$F(x,y,z) = x^2 - 2(y^2+z^2)x - (y^3+yz+z^3).$$

The roots of $F(x,y,z)$ w.r.t. x are

$$\chi_i(y,z) = y^2+z^2-(-1)^i\{(y^2+z^2)^2+(y^3+yz+z^3)\}^{1/2}, \quad i=1,2.$$

By defining the weights of y and z in any way, series expansion of the root $\chi_i(y,z)$ is classified into one of the following five cases.

Case 1. weights $= (y:1,\ z:\infty)$: expressed with $\sqrt{y^3}$,

Case 2. weights $= (y:1,\ z:2)$: expressed with $\sqrt{y^3+yz}$,

Case 3. weights $= (y:1,\ z:1)$: expressed with $\sqrt{yz}$,

Case 4. weights $= (y:2,\ z:1)$: expressed with $\sqrt{z^3+zy}$,

Case 5. weights $= (y:\infty,\ z:1)$: expressed with $\sqrt{z^3}$.

In fact, the series expansions become as follows (since $F(x,y,z)$ is symmetric w.r.t. y and z, we consider only Cases $1 \sim 3$).

Case 1: $\chi_i = (y^2+z^2)-(-1)^i\sqrt{y^3}\Big\{1+\frac{1}{2}R_1(y,z)-\frac{1}{8}R_1^2(y,z)+\cdots\Big\}$,
where $R_1(y,z) = (yz+z^3+y^4+2y^2z^2+z^4)/y^3$,

Case 2: $\chi_i = (y^2+z^2)-(-1)^i\sqrt{y^3+yz}\Big\{1+\frac{1}{2}R_2(y,z)-\frac{1}{8}R_2^2(y,z)+\cdots\Big\}$,
where $R_2(y,z) = (z^3+y^4+2y^2z^2+z^4)/(y^3+yz)$,

Case 3: $\chi_i = (y^2+z^2)-(-1)^i\sqrt{yz}\Big\{1+\frac{1}{2}R_3(y,z)-\frac{1}{8}R_3^2(y,z)+\cdots\Big\}$,
where $R_3(y,z) = (y^3+z^3+y^4+2y^2z^2+z^4)/yz$.

We see that all of these expressions show different analytic behaviors at $y=0$ and $z=0$. □

4.2 Extended Hensel construction w.r.t. the total-degree

Although property (4.8) is not always satisfied, expansion w.r.t. the total-degree seems to be most desirable. Therefore, in this subsection, we investigate it in details. Note that the expansions w.r.t. total-degree in previous subsection were obtained through the root formula hence the method is not applicable to high degree polynomials.

We introduce the total-degree variable t by the replacement $y \to ty,\ \ldots,\ z \to tz$ in $F(x,y,\ldots,z)$, hence the total-degree is for subvariables $y,\ldots,z$.

Definition 3 (Newton's line for multivariate polynomial) *For each nonzero term $cx^{e_x}y^{e_y}\cdots z^{e_z}$ of $F(x,y,\ldots,z)$, we plot a dot at the point $(e_x, e_y+\cdots+e_z)$ in the e_x-e_t two-dimensional Cartesian coordinate system. Let L be a straight line such that it passes the point $(d,0)$ as well as another dot plotted and that no dot plotted is below L. The line L is called Newton's line for F.*

Definition 4 (Newton's polynomial for multivariate polynomial) *The sum of all the terms of $F(x,y,\ldots,z)$, which are plotted on Newton's line is called Newton's polynomial for F.*

Let Newton's line be $e_x/d + e_t/\delta = 1$. Let $\hat\delta$ and $\hat d$ be positive integers satisfying

$$\hat\delta/\hat d = \delta/d, \quad \gcd(\hat\delta,\hat d) = 1. \tag{4.9}$$

We define ideals S_k $(k=1,2,\ldots)$ as follows.

$$\begin{aligned} S_k &= (x, t^{\hat\delta/\hat d})^d \times (t^{1/\hat d})^k \\ &= (x^d t^{(k+0)/\hat d},\ x^{d-1}t^{(k+\hat\delta)/\hat d},\ x^{d-2}t^{(k+2\hat\delta)/\hat d},\ \ldots,\ x^0 t^{(k+d\hat\delta)/\hat d}). \end{aligned} \tag{4.10}$$

Remark 5 Since a term $cx^{e_x}y^{e_y}\cdots z^{e_z}$ is plotted uniquely at the point $(e_x, e_y+\cdots+e_z)$ in the e_x-e_t plane, Lemma 1 in **3** tells us that any term of $F(x,y,\ldots,z)$ is plotted at one of the points

$$\Big\{(d-0,(k+0)/\hat d),\ (d-1,(k+\hat\delta)/\hat d),\ \ldots,\ (0,(k+d\hat\delta)/\hat d)\ \Big|\ k=0,1,2,\ldots\Big\}.$$

Remark 6 The total-degree variable t will appear only temporally in the computation and it will be erased at the last step of computation. Therefore, we will say, for example, that $F(x,y,\ldots,z)$ with total-degree variable is in $\mathbf{C}[x,y,\ldots,z]$. However, we will also say, for example, that $F^{(0)}(x,y,\ldots,z)$ is a homogeneous polynomial in x and $t^{\hat\delta/\hat d}$. Note that there is a good similarity between the treatment for bivariate case and that for multivariate case when viewed w.r.t. variables x and t.

In the bivariate case, we have seen that $F(x,y)$ can be factorized in $\mathbf{C}\{y\}[x]$ if corresponding Newton's polynomial $F^{(0)}(x,y)$ can be factorized in $\mathbf{C}[x,y]$ (Theorem 2). In this subsection, we show that a similar factorization can be performed for multivariate polynomial $F(x,y,\ldots,z)$, too. Let $F^{(0)}(x,y,\ldots,z)$ be Newton's polynomial for $F(x,y,\ldots,z)$, defined as above, then $F^{(0)}$ is homogeneous w.r.t. x and $\hat t = t^{\hat\delta/\hat d}$. In general, $F^{(0)}$ can be factorized in $\mathbf{C}[x,y,\ldots,z]$ as

$$F^{(0)}(x,y,\ldots,z) = H_1(x,y,\ldots,z)^{m_1}\cdots H_s(x,y,\ldots,z)^{m_s} \tag{4.11}$$

(irreducible factorization in $\mathbf{C}[x,y,\ldots,z]$),

$$\gcd(H_i,H_j) = 1 \ \text{ for any } \ i \neq j, \tag{4.12}$$

$$\deg_x(H_i) = r_i, \quad i=1,\ldots,s. \tag{4.13}$$

Suppose that $s \geq 2$. We put

$$F_i^{(0)}(x,y,\ldots,z) = H_i(x,y,\ldots,z)^{m_i}, \quad i=1,\ldots,s. \tag{4.14}$$

Lemma 5 *For each $l \in \{0, \ldots, d-1\}$, there exist Lagrange's interpolation polynomials $W_1^{(l)}, \ldots, W_s^{(l)}$ in $K(y, \ldots, z)[x]$, satisfying*

$$\begin{aligned} &W_1^{(l)}[F_1^{(0)} \cdots F_s^{(0)}/F_1^{(0)}] + \cdots + W_s^{(l)}[F_1^{(0)} \cdots F_s^{(0)}/F_s^{(0)}] = x^l, \\ &\deg_x(W_i^{(l)}) < \deg_x(F_i^{(0)}), \quad i = 1, \ldots, s. \end{aligned} \tag{4.15}$$

For each $i \in \{1, \ldots, s\}$, $W_i^{(l)}\hat{t}^{d-l}$ $(l=0, \ldots, d-1)$, with $\hat{t} = t^{\hat{\delta}/\hat{d}}$, are homogeneous polynomials in x and $\hat{t}$, of total-degree $\deg_x(F_i^{(0)})$ w.r.t. x and $\hat{t}$.

Proof. We can calculate $W_i^{(l)}$ by the extended Euclidean algorithm which consists of only rational operations on the coefficients w.r.t. x. Hence, $W_i^{(l)} \in K(y, \ldots, z)[x]$. Since $F_i^{(0)}$ is homogeneous w.r.t. x and $\hat{t}$, so is $W_i^{(l)}$. Since $\deg_x(W_i^{(l)}[F_1^{(0)} \cdots F_s^{(0)}/F_i^{(0)}]) < d$, if we multiply $\hat{t}^{d-l}$ to the upper equality in (4.15), both sides of the equality become homogeneous polynomials w.r.t. x and $\hat{t}$. Therefore, $W_i^{(l)}\hat{t}^{d-l}$ is a homogeneous polynomial in x and $\hat{t}$. □

Remark 7 Contrary to (3.9) for bivariate case, we defined Lagrange's interpolation polynomials without multiplying $\hat{t}^{d-l}$ to the upper equality in (4.15). The reason is that, in the multivariate case, $W_1^{(l)}, \ldots, W_s^{(l)}$ are rational functions in $y, \ldots, z$ with different denominators in general. Hence, we cannot clear the denominators by multiplying a simple expression to the equality.

Theorem 4 *Let $F(x, y, \ldots, z)$ be a polynomial in $K[x, y, \ldots, z]$, monic w.r.t. x and square-free. Let $F^{(0)}(x, y, \ldots, z)$ be Newton's polynomial for F, defined as in* Def. 4, *and be factorized in $K[x, y, \ldots, z]$ as in* (4.11) *with* (4.12). *Then, for any positive integer k, we can construct $F_i^{(k)}(x, y, \ldots, z) \in K(y, \ldots, z)[x]$ $(i=1, \ldots, s)$, satisfying*

$$\begin{aligned} &F(x, y, \ldots, z) \equiv F_1^{(k)}(x, y, \ldots, z) \cdots F_s^{(k)}(x, y, \ldots, z) \pmod{S_{k+1}}, \\ &F_i^{(k)}(x, y, \ldots, z) \equiv H_i(x, y, \ldots, z)^{m_i} \pmod{S_1}, \quad i = 1, \ldots, s. \end{aligned} \tag{4.16}$$

Furthermore, $F_1^{(k)}, \ldots, F_s^{(k)}$ are polynomials in total-degree variable t.

Proof. Since $\gcd(F_i^{(0)}, F_j^{(0)}) = 1$ for any $i \neq j$ and we have Lagrange's interpolation polynomials $W_1^{(l)}, \ldots, W_s^{(l)}$ $(l = 1, \ldots, d-1)$, we can perform the extended Hensel construction with moduli S_k $(k = 1, 2, \ldots)$. Then, the theorem can be proved similarly as Theorem 2. □

Corollary 2 *Each coefficient of term $x^{e_x}t^{e_t}$ in $F_i^{(k)}$ is of the form N/D, where N and D are homogeneous polynomials in $y, \ldots, z$ and* $\mathrm{tdeg}(N) - \mathrm{tdeg}(D) = e_t$.

Example 6 Extended Hensel construction in $\mathbf{C}(y, z)[x]$.

$$F(x, y, z) = x^3 + (y - z + z^2)x^2 - (y + z + y^2 - z^2)x + (y^2 - z^3).$$

Introducing the total-degree variable t, we find Newton's polynomial $F^{(0)} = x^3 - t(y+z)x$. We see $\hat{d} = 2$ and $\hat{\delta} = 1$, hence $\hat{t} = t^{1/2}$.

$F^{(0)}$ is factorized in $\mathbf{C}[x,y,z]$ as follows.

$$F^{(0)}(x,y,z) = x \times [x^2 - t(y+z)].$$

We put $F_1^{(0)} = x$ and $F_2^{(0)} = x^2 - t(y+z)$. Lagrange's interpolation polynomials $W_1^{(l)}, W_2^{(l)}$ $(l=0,1,2)$ are calculated as follows.

$$\begin{array}{ll} W_1^{(0)} = \dfrac{-1}{t(y+z)} \ , & W_2^{(0)} = \dfrac{x}{t(y+z)} \ , \\ W_1^{(1)} = 0 \ , & W_2^{(1)} = 1 \ , \\ W_1^{(2)} = 0 \ , & W_2^{(2)} = x \ . \end{array}$$

Note that $W_1^{(0)}\hat{t}^{d-0} \propto \hat{t}$ and $W_2^{(0)}\hat{t}^{d-0} \propto x\hat{t}$, $W_2^{(1)}\hat{t}^{d-1} \propto \hat{t}^2$, $W_2^{(2)}\hat{t}^{d-2} \propto x\hat{t}$, in consistent with the claim on degree of $W_i^{(l)}\hat{t}^{d-l}$ in Lemma 5.

Ideals S_k $(k=1,2,\ldots)$ are given by

$$S_k = (x, t^{1/2})^3 \times (t^{1/2})^k = (x^3 t^{k/2},\ x^2 t^{(k+1)/2},\ x t^{(k+2)/2},\ t^{(k+3)/2}).$$

Performing the Extended Hensel construction by one step, we obtain

$$\begin{array}{rcll} \mathit{\Delta} F^{(1)} & \equiv & F - F_1^{(0)}F_2^{(0)} \pmod{S_2} & = \ t(y-z)\,x^2 + 0\,x + t^2y^2. \\ F_1^{(1)} & = & F_1^{(0)} + W_1^{(2)}t(y-z) + W_1^{(0)}t^2y^2 & = \ x - t\,\dfrac{y^2}{y+z}, \\ F_2^{(1)} & = & F_2^{(0)} + W_2^{(2)}t(y-z) + W_2^{(0)}t^2y^2 & = \ x^2 - t(y+z) + xt\,\dfrac{2y^2-z^2}{y+z}. \end{array}$$

Performing one more construction, we obtain

$$\begin{array}{rcl} \mathit{\Delta} F^{(2)} & \equiv & F - F_1^{(1)}F_2^{(2)} \ \equiv \ xt^2R_2(y,z) \pmod{S_3}, \\ & & \text{where} \ \ R_2(y,z) = \dfrac{y^4 - 2y^3z - y^2z^2 + 2yz^3 - z^4}{y^2+2yz+z^2}, \end{array}$$

$$\begin{array}{rcll} F_1^{(2)} & = & F_1^{(1)} + W_1^{(1)}t^2R_2(y,z) & = \ F_1^{(1)} = x - t\,\dfrac{y^2}{y+z}, \\ F_2^{(2)} & = & F_2^{(1)} + W_2^{(1)}t^2R_2(y,z) & = \ x^2 - t(y+z) + xt\,\dfrac{2y^2-z^2}{y+z} + xt^2R_2(y,z). \end{array}$$

Note that $F_1^{(2)}$ and $F_2^{(2)}$ are polynomials in t with coefficients of rational functions in $y,\ldots,z$, as Theorem 4 claims. $\square$

4.3 Introduction of algebraic functions

In this subsection, we consider factorizing $F_i^{(k)}(x, y, \ldots, z)$ $(i=1,\ldots,s)$ in (4.16) into fractional-power series w.r.t. total-degree variable t. This factorization will be performed similarly as described in **3.2**, by introducing algebraic functions $\theta_1,\ldots,\theta_r$ the minimal polynomial of which is much simpler than F. Since this factorization is the same for each $F_i^{(k)}$, $1 \le i \le s$, we redefine $F(x,y,\ldots,z)$ as follows.

$$\begin{aligned} &F(x,y,\ldots,z) \in \mathbf{C}(y,\ldots,z)[x], \\ &F(x,y,\ldots,z) \equiv H(x,y,\ldots,z)^m \pmod{S_1}, \end{aligned} \tag{4.17}$$

where the modulus S_1 is defined by (4.10), and

$$\begin{aligned} &H(x,y,\ldots,z) \in \mathbf{C}[x,y,\ldots,z], \\ &H(0,y,\ldots,z) \neq 0, \quad \deg_x(H) = r \ge 2. \end{aligned} \tag{4.18}$$

(That is, we exclude the case of $H = x^m$.) Furthermore, H is monic w.r.t. x, irreducible over $\mathbf{C}$ hence square-free, and homogeneous w.r.t. x and $\hat{t} = t^{\hat{\delta}/\hat{d}}$.

Let the roots of $H(x,y,\ldots,z)\,|_{t=1}$ w.r.t. x be $\theta_1,\ldots,\theta_r$, hence

$$H(x,y,\ldots,z) = (x - \hat{t}\theta_1(y,\ldots,z)) \cdots (x - \hat{t}\theta_r(y,\ldots,z)). \tag{4.19}$$

Note that $\theta_i \neq \theta_j$ for any $i \neq j$. The θ_i in (4.19) corresponds to ζ_i in (3.6). Contrary to the bivariate case where ζ_i may be either an algebraic number or a complex number computed approximately, we must introduce algebraic functions $\theta_1,\ldots,\theta_r$, with minimal polynomial $H(x,y,\ldots,z)$, in the multivariate case. That is, we treat an algebraic extension field $\mathbf{C}(y,\ldots,z)(\theta_1,\ldots,\theta_r)$. We put

$$G_i^{(0)}(x,y,\ldots,z) = (x - \hat{t}\theta_i(y,\ldots,z))^m, \quad i = 1,\ldots,r. \tag{4.20}$$

Lemma 6 *For each $l \in \{0,\ldots,mr-1\}$, there exist Lagrange's interpolation polynomials $W_1^{(l)},\ldots,W_r^{(l)}$ which are polynomials in x, satisfying*

$$\begin{aligned} &W_1^{(l)}[G_1^{(0)} \cdots G_r^{(0)}/G_1^{(0)}] + \cdots + W_r^{(l)}[G_1^{(0)} \cdots G_r^{(0)}/G_r^{(0)}] = x^l, \\ &\deg_x(W_i^{(l)}(x,y,\ldots,z)) < m, \quad i = 1,\ldots,r. \end{aligned} \tag{4.21}$$

Each $W_i^{(l)}$, $1 \le i \le r$, can be expressed as

$$\begin{aligned} W_i^{(l)}(x,y,\ldots,z) &= w_{r-1}^{(l)}(x,y,\ldots,z)(\hat{t}\theta_i)^{r-1} + \cdots + w_0^{(l)}(x,y,\ldots,z)(\hat{t}\theta_i)^0, \\ w_j^{(l)}(x,y,\ldots,z) &\in \mathbf{C}(y,\ldots,z)[x], \quad j = 0,\ldots,r-1. \end{aligned} \tag{4.22}$$

(That is, $w_j^{(l)}$ $(j=0,\ldots,r-1)$ are independent of index i.) Furthermore, $W_i^{(l)}\hat{t}^{mr-l}$ is a homogeneous polynomial in x and $\hat{t}$.

Proof. We can calculate $W_i^{(l)}$ $(i=1,\ldots,r)$ satisfying (4.21) by the extended Euclidean algorithm, therefore the existence of $W_i^{(l)}$ is assured.

For each $l \in \{0,\ldots,mr-1\}$, we express $W_i^{(l)}$ as

$$W_i^{(l)} = \tilde{w}_{i,m-1}^{(l)}(t,y,\ldots,z)x^{m-1} + \cdots + \tilde{w}_{i,0}^{(l)}(t,y,\ldots,z)x^0, \quad i = 1,\ldots,r,$$

and determine the coefficients $\tilde{w}_{i,j}^{(l)}$ $(j = m-1,\ldots,0)$ as follows. For $\mu = 0,\ldots,m-1$, we calculate the μth derivative w.r.t. x, of both sides of upper equality in (4.21). This gives us m equations. Then, for each $i \in \{1, \dots, r\}$, we substitute $\hat{t}\theta_i$ for x in these m equations. This gives us a system of mr equations which are linear w.r.t. $\tilde{w}_{i,j}^{(l)}$. By solving this system, we can determine $\tilde{w}_{i,j}^{(l)}$ $(i = 1, \dots, r,\ j = 0, \dots, m-1)$. Note that, since $H(x, y, \dots, z)$ is monic w.r.t. x, irreducible over $\mathbf{C}$ and $H(\hat{t}\theta_j, y, \dots, z) = 0$, we can perform these calculations modulo $H(\hat{t}\theta_j, y, \dots, z)$ $(j=1, \dots, r)$. Thus, $W_i^{(l)}$ is determined to be a polynomial in x and $\hat{t}\theta_j$ $(j=1, \dots, r)$, with coefficients which are rational functions in $y, \dots, z$, satisfying $\deg_x(W_i^{(l)}) < m$ and $\deg_{\theta_j}(W_i^{(l)}) < r$.

Substituting $\hat{t}\theta_i$ for x in the μth derivative of upper equality in (4.21), we obtain

$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}[W_i^{(l)} G_1^{(0)} \cdots G_r^{(0)} / G_i^{(0)}] \Big|_{x=\hat{t}\theta_i} = \frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} x^l \Big|_{x=\hat{t}\theta_i} \tag{4.23}$$
$$\text{(for } \mu = 0, \text{ no differentiation is made above)},$$

because other terms contain factor $(x - \hat{t}\theta_i)$ hence vanish by the substitution $x \to \hat{t}\theta_i$. Thus, $W_i^{(l)}$ is determined by (4.23) only, with $\mu=0, \dots, m-1$. That is, we can express $W_i^{(l)}$ in terms of $\hat{t}\theta_i$ only, without using $\hat{t}\theta_j$ $(j \neq i)$. Furthermore, since

$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu}[G_1^{(0)} \cdots G_r^{(0)} / G_i^{(0)}] \Big|_{x=\hat{t}\theta_i} = \frac{1}{m!} \frac{\mathrm{d}^{m+\mu}}{\mathrm{d}x^{m+\mu}} [G_1^{(0)} \cdots G_r^{(0)}] \Big|_{x=\hat{t}\theta_i},$$

and $G_1^{(0)} \cdots G_r^{(0)} = H(x, y, \dots, z)^m$, the linear system derived from (4.23), with $\mu = 0, \dots, m-1$, is the same form for all the θ_i $(i=1, \dots, r)$. Therefore, $W_i^{(l)}$ can be expressed as in (4.22).

That $W_i^{(l)} \hat{t}^{mr-l}$ is a homogeneous polynomial in x and $\hat{t}$ is a direct consequence of that $G_j^{(0)}$ $(j=1, \dots, r)$ are so. $\square$

Example 7 Calculation of Lagrange's interpolation "polynomials".

$$F(x, y, z) = x^3 - (y - z)x^2 + (y + 2z + y^2 - 2z^2)x - (y + z - y^2 - z^2).$$

Newton's polynomial for F is $F^{(0)} = x^3 - t(y + z)$. Hence

$$H(x, y, z) = x^3 - t(y + z) = (x - \hat{t}\theta_1)(x - \hat{t}\theta_2)(x - \hat{t}\theta_3),$$

where $\hat{t} = t^{1/3}$, $\theta_1 = \theta$, $\theta_2 = \omega\theta$ and $\theta_3 = \omega^2\theta$, with $\theta = \sqrt[3]{y + z}$ and $\omega = (-1 + \sqrt{3}i)/2$, a primitive cube root of 1. We put

$$G_i^{(0)}(x, y, z) = x - \hat{t}\theta_i, \quad i = 1, 2, 3.$$

Following the method described in the above proof, we calculate $W_i^{(l)}$ $(l=0, 1, 2)$. Note that, since $m = 1$, $W_i^{(l)}$ is constant w.r.t. x and we have the case $\mu = 0$ only. Thus, we have only to solve the following single equation to calculate $W_i^{(l)}$:

$$W_i^{(l)} \, \mathrm{d}H/\mathrm{d}x \Big|_{x=\hat{t}\theta_i} = x^l \Big|_{x=\hat{t}\theta_i}.$$

Since $(\hat{t}\theta_i)^3 - t(y+z) = 0$, this equation gives us

$$W_i^{(l)} = \frac{(\hat{t}\theta_i)^l}{3(\hat{t}\theta_i)^2} = \frac{(\hat{t}\theta_i)^{l+1}}{3t(y+z)}.$$

If $l+1 \geq 3$ the numerator can be reduced further. Thus, we obtain

$$W_i^{(0)} = \frac{\hat{t}\theta_i}{3t(y+z)}, \qquad W_i^{(1)} = \frac{(\hat{t}\theta_i)^2}{3t(y+z)}, \qquad W_i^{(2)} = \frac{1}{3}.$$

Using relations $\theta_1+\theta_2+\theta_3 = 0$, $\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1 = 0$, and $\theta_1\theta_2\theta_3 = y+z$, we can easily confirm that the above expressions satisfy (4.21). □

Theorem 5 *Let the extended Hensel construction of* $F(x,y,\ldots,z)$ *in* (4.17), *with initial factors* $G_i^{(0)}(x,y,\ldots,z)$ $(i=1,\ldots,r)$ *given in* (4.20) *and moduli* S_k $(k=1,2,\ldots)$ *given in* (4.10), *be*

$$\begin{aligned} &F(x,y,\ldots,z) \equiv G_1^{(k)}(x,y,\ldots,z)\cdots G_r^{(k)}(x,y,\ldots,z) \pmod{S_{k+1}}, \\ &G_i^{(k)}(x,y,\ldots,z) \equiv G_i^{(0)}(x,y,\ldots,z) \pmod{S_1}, \quad i=1,\ldots,r. \end{aligned} \tag{4.24}$$

Then, $G_i^{(k)}$ $(i=1,\ldots,r)$ *can be expressed as*

$$\begin{aligned} &G_i^{(k)}(x,y,\ldots,z) = g_{r-1}^{(k)}(x,y,\ldots,z)(\hat{t}\theta_i)^{r-1} + \cdots + g_0^{(k)}(x,y,\ldots,z)(\hat{t}\theta_i)^0, \\ &g_j^{(k)}(x,y,\ldots,z) \in \mathbf{C}(y,\ldots,z)[x], \quad j=0,\ldots,r-1. \end{aligned} \tag{4.25}$$

(That is, $g_j^{(k)}$ $(j=0,\ldots,r-1)$ *are independent of index* i*.)*

Proof. We note that, using Lagrange's interpolation polynomials $W_i^{(l)}$ defined in (4.21), we can perform the extended Hensel construction as expressed in (4.24).

Now, the theorem is apparently valid for $k=0$. Suppose the theorem is valid up to the $(k-1)$-st construction step $(k \geq 1)$. According to the procedure of Hensel construction, $G_i^{(k)}$ $(i=1,\ldots,r)$ are calculated as follows: put $G_i^{(k)} = G_i^{(k-1)} + \Delta G_i^{(k)}$, $\Delta G_i^{(k)} \equiv 0 \pmod{S_k}$, then calculate

$$\begin{aligned} \Delta F^{(k)} &\equiv F - G_1^{(k-1)}\cdots G_r^{(k-1)} \pmod{S_{k+1}} \\ &\equiv f_{mr-1}^{(k)}(y,\ldots,z)x^{mr-1} + \cdots + f_0^{(k)}(y,\ldots,z)x^0, \end{aligned}$$

and calculate $\Delta G_i^{(k)}$ as

$$\Delta G_i^{(k)} = \sum_{l=0}^{mr-1} W_i^{(l)}(x,y,\ldots,z) f_l^{(k)}(y,\ldots,z).$$

By the induction assumption, $f_l^{(k)}(y,\ldots,z)$ $(l=0,\ldots,mr-1)$ are symmetric w.r.t. $\hat{t}\theta_1,\ldots,\hat{t}\theta_r$, hence we can express $f_l^{(k)}(y,\ldots,z)$ without using $\hat{t}\theta_1,\ldots,\hat{t}\theta_r$. Then, Lemma 6 tells us that $G_i^{(k)}$ can be expressed as in (4.25). □

Example 8 Fractional-power series expansion of multivariate roots.

$$F(x,y,z) = x^3 - (y-z)x^2 + (y+2z+y^2-2z^2)x - (y+z-y^2-z^2).$$

This is the same polynomial as used in Example 7, hence we use the results in Example 7 and show only the Hensel construction step with moduli

$$S_k = (x, t^{1/3})^3 \times (t^{1/3})^k = (x^3 t^{k/3},\ x^2 t^{(k+1)/3},\ x t^{(k+2)/3},\ t^{(k+3)/3}).$$

We put $\hat{t} = t^{1/3}$ and $F_i^{(0)} = x - \hat{t}\theta_i$ $(i=1,2,3)$. Then, we obtain

$$\begin{aligned}
\Delta F^{(1)} &\equiv F - F_1^{(0)} F_2^{(0)} F_3^{(0)} &&\equiv t(y+2z)x \pmod{S_2},\\
F_i^{(1)} &= F_i^{(0)} + W_i^{(1)} t(y+2z) &&= x - (\hat{t}\theta_i) + \frac{y+2z}{3(y+z)}(\hat{t}\theta_i)^2.
\end{aligned}$$

We can confirm easily that the above construction is the same for $i = 1, 2$ and 3. We show one more construction, where the calculation is done by using relations $\theta_1+\theta_2+\theta_3 = 0$, $\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1 = 0$, $\theta_1\theta_2\theta_3 = y+z$, and $\theta_1^3 = \theta_2^3 = \theta_3^3 = y+z$.

$$\begin{aligned}
\Delta F^{(2)} &\equiv F - F_1^{(1)} F_2^{(1)} F_3^{(1)} &&\equiv -t(y-z)x^2 \pmod{S_3},\\
F_i^{(2)} &= F_i^{(1)} - W_i^{(2)} t(y-z) &&= x - (\hat{t}\theta_i) + \frac{y+2z}{3(y+z)}(\hat{t}\theta_i)^2 - \frac{t(y-z)}{3}.
\end{aligned}$$

In this way, we can construct $F_i^{(k)}$, $k=1,2,3,\ldots$, as Theorem 5 claims. □

4.4 Calculating the multivariate roots $\chi(y,\ldots,z)$

In this subsection, we consider factorizing $F(x,y,\ldots,z)$ into linear factors as

$$F(x,y,\ldots,z) = (x-\chi_1(y,\ldots,z))\cdots(x-\chi_d(y,\ldots,z)). \tag{4.26}$$

Note that, although Example 8 above shows a calculation of multivariate roots, it is a special case of $m = 1$ in (4.17). In order to calculate the roots of $F(x,y,\ldots,z)$ in general case, we must consider the case of $m \geq 2$ in (4.17). That is, we must consider calculating the roots of $G(x,y,\ldots,z)$ such that

$$\begin{aligned}
&G(x,y,\ldots,z) \in \mathbf{C}(y,\ldots,z)(\theta)[x], \quad \hat{t} = t^{\hat{\delta}/\hat{d}},\\
&G(x,y,\ldots,z) \equiv (x-\hat{t}\theta)^m \pmod{S_1}, \quad m \geq 2,
\end{aligned} \tag{4.27}$$

where $\hat{t}\theta$ is a root of monic irreducible polynomial $H(x,y,\ldots,z)$ which is homogeneous w.r.t. x and $\hat{t}$, and S_1 is defined in (4.10).

We will calculate the roots of $G(x,y,\ldots,z)$ similarly as explained in **3.3**. That is, we first perform the following transformation for G:

$$G(x,y,\ldots,z) \Longrightarrow G'(x,y,\ldots,z) = G(x+\hat{t}\theta,y,\ldots,z). \tag{4.28}$$

Then, renaming G' as G for simplicity, we have

$$\begin{aligned} &G(x,y,\ldots,z) \in \mathbf{C}(y,\ldots,z)(\theta)[x], \\ &G(x,y,\ldots,z) \equiv x^m \pmod{S_1}, \quad m \geq 2, \end{aligned} \tag{4.29}$$

and we encounter a problem which is similar to that treated in the previous subsections. Only one difference is that $G(x,y,\ldots,z)$ is now a polynomial in x and $\hat{t}$ over the algebraic extension field $\mathbf{C}(y,\ldots,z)(\theta)$.

For $G(x,y,\ldots,z)$, we can draw Newton's line and calculate Newton's polynomial $G^{(0)}(x,y,\ldots,z)$, as defined in **4.2**, because G is a polynomial in x and $\hat{t}$. Furthermore, we can define ideals $\bar{S}_k$ $(k=1,2,\ldots)$ as in (3.32), because $\bar{S}_k$ is defined by only two variables x and $\hat{t}$. Then, we first factorize $G^{(0)}(x,y,\ldots,z)$ in $\mathbf{C}(y,\ldots,z)(\theta)[x]$, as we have factorized $F^{(0)}(x,y,\ldots,z)$ as in (4.11), then factorize each factor of $G^{(0)}$ "formally" by introducing algebraic functions $\Theta_1,\ldots,\Theta_\rho$, as we have introduced $\theta_1,\ldots,\theta_r$ in (4.19). That is, we treat an algebraic extension field $\mathbf{C}(y,\ldots,z)(\theta,\Theta_1,\ldots,\Theta_\rho)$ as coefficient domain of factor polynomials. Since the Euclidean algorithm works over this extension field, we can calculate Lagrange's interpolation polynomials and perform the extended Hensel construction for $G(x,y,\ldots,z)$ with moduli $\bar{S}_k$ $(k=1,2,\ldots)$. Suppose that $G(x,y,\ldots,z)$ is factorized by this construction as

$$G(x,y,\ldots,z) \equiv G_1^{(k)}(x,y,\ldots,z)\cdots G_\rho^{(k)}(x,y,\ldots,z) \pmod{\bar{S}_{k+1}}, \tag{4.30}$$

then, as we have proved Theorem 5, $G_i^{(k)}, 1 \leq i \leq \rho$, is expressed as a polynomial w.r.t. θ and Θ_i only, with coefficients of rational functions in $y,\ldots,z$, where the coefficients are independent of index i.

This process can be repeated, and we have the following theorem.

Theorem 6 *Let $F(x,y,\ldots,z) \in K[x,y,\ldots,z]$ be monic w.r.t. x and square-free. Then, by repeated use of the extended Hensel construction, we can factorize $F(x,y,\ldots,z)$ into the form* (4.26). *Let $\Theta^{(1)},\ldots,\Theta^{(\sigma)}$ be algebraic functions, where Θ_1 $(=\theta)$ is an algebraic function to be introduced first by the procedure mentioned above, with a minimal polynomial $H^{(1)}(x,y,\ldots,z)$ over $\mathbf{C}(y,\ldots,z)$, and $\Theta^{(j)}$ $(2 \leq j \leq \sigma)$, is an algebraic function to be introduced j-th, with a minimal polynomial $H^{(j)}(x,y,\ldots,z)$ over $\mathbf{C}(y,\ldots,z)(\Theta^{(1)},\ldots,\Theta^{(j-1)})$. Then, each root $\chi(y,\ldots,z)$ of $F(x,y,\ldots,z)$, w.r.t. x is expressed as a polynomial w.r.t. $\Theta^{(1)},\ldots,\Theta^{(\sigma)}$, where each coefficient is of the form N/D, with N and D homogeneous polynomials in $y,\ldots,z$ and* $\mathrm{tdeg}(N) - \mathrm{tdeg}(D) \geq 0$.

Proof. Repeated applicability of the extended Hensel construction and the form of each factor of F have been proved above, so we have only to prove that we obtain linear factors finally. Suppose that we cannot factorize F into linear factors, then the above discussions mean that F contains a factor of the form $(x-\chi)^m$, $m \geq 2$, which contradicts that F is square-free. $\square$

5 Concluding Remarks

We have shown in this paper that a slight extension of the generalized Hensel construction allows us to calculate the roots of multivariate polynomial $F(x,y,\ldots,z)$ in the same way as for bivariate polynomial $F(x, y)$. The roots in multivariate case are expressed as fractional-power series w.r.t. the total-degree, by using algebraic functions $\Theta^{(1)}, \ldots, \Theta^{(\sigma)}$ whose minimal polynomials are much simpler than the given polynomial. As we have pointed in Remark 6, there is a good similarity between theories for bivariate and multivariate cases, when viewed w.r.t. the main variable x and the total-degree variable t. This fact allows us to treat the multivariate case simply.

For bivariate polynomial $F(x, y)$, one can use our method in two ways: one may treat the roots $\zeta_1, \ldots, \zeta_r$ of Newton's polynomial $F^{(0)}(x, y)$ as algebraic numbers, or one may calculate $\zeta_1, \ldots, \zeta_r$ numerically hence approximately.

When $\zeta_1, \ldots, \zeta_r$ are treated as algebraic numbers, the number of algebraic numbers to be introduced by our method to represent a root will be much smaller than that by Newton-Puiseux's method. The reason is that our method introduces algebraic numbers systematically for all the roots and each root is expressed simply as in Theorem 5 which is valid in bivariate case, too. In this point, Kung-Traub's method is not much different from Newton-Puiseux's method, because Kung-Traub's method calculates each root separately by employing the Newton polygon to determine the algebraic numbers, as Newton-Puiseux's method does.

When $\zeta_1, \ldots, \zeta_r$ are calculated approximately, our method will be much more useful than Newton-Puiseux's method in that it is pretty safe against the numerical errors. If a branch point P_B is determined approximately as $P = P_B + \delta P$, where P is an approximate branch point with an error δP, then Newton-Puiseux's method often gives the Taylor series expansion at P instead of Puiseux series expansion at P_B. Hence, Newton-Puiseux's method is quit sensitive to the numerical errors. Our method is composed of three steps, the first is to determine Newton's polynomial and calculate Lagrange's interpolation polynomials. The second is to select terms by the ideal S_{k+1} and perform the extended Hensel construction. The third is to perform the transformation $G(x, y) \Longrightarrow G(x + \zeta\hat{t}, y)$. Most part of the computation is included in the second step which is safe against numerical errors, because necessary terms are selected by the ideal S_{k+1} definitely. Furthermore, other two steps are not so dangerous, although numerical accuracy may be lost by cancellation of almost the same numbers. In fact, Shiihara and Sasaki [SS96] applied our method with floating-point number arithmetic to analytic continuation and Riemann surface determination of algebraic functions successfully.

When approximate arithmetic is employed, factorization in (4.11) may be not exact but approximate. For such *approximate factorization*, conventional factorization algorithm breaks down but we can apply approximate factorization algorithm by Sasaki et al. [SSKS91].

One may think that our method is very complicated when applied to multivariate polynomials, but this is not true: our method can be executed rather simply, as Examples $6 \sim 8$ show. We have not considered application of our method for multivariate polynomials. We think, however, that if algebraic functions θ and Θ are expressed explicitly by radicals, such as $\theta = \sqrt[3]{y + z}$, then the roots obtained by our method will be quite useful; they show analytic behaviors of algebraic functions $\chi_i(y, \ldots, z)$ $(i = 1, \ldots, d)$ around a singular point rather well.

In order to apply our method, as well as methods for regular case, to many practical problems, many investigations are necessary. In particular, error analysis of the methods with floating-point number arithmetic is very important, and we have performed such an analysis partially for multivariate power series expansion of the roots of $F(x, y, \ldots, z)$, see [SKK94].

References

[Abe73] O. Aberth, *Iterative method for finding all zeros of a polynomial simultaneously.* Math. Comp. **27** (1973), 339-344.

[Dur60] E. Durand, *Solutions numeriques des equations algébriques Tom 1 : equation du tupe* $F(x) = 0$. Racines d'um Polnome Masson, Paris, 1960.

[Duv89] D. Duval, *Rational Puiseux expansions.* Compos. Math. **70** (1989), 119-154.

[GCL92] K. O. Geddes, S. R. Czapor and G. Labahn, *Algorithms for Computer Algebra*, Kluwer Academic Pub., Boston-Dordrecht-London, 1992.

[Hen08] K. Hensel, *Theorie der algebraischen Zahlen.* Teubner, Berlin, 1908.

[Iri81] M. Iri, *Numerical computation* (in Japanese). Asakura Shoten, Tokyo, 1981.

[Ker66] I. O. Kerner, *Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polnomen.* Numer. Math. **8** (1966), 290-294.

[KT78] H. T. Kung and J. F. Traub, *All algebraic functions can be computed fast.* J. ACM **25** (1978), 245-260.

[Lau82] M. Lauer, *Computing by homomorphic images.* In: Computer Algebra, Symbolic and Algebraic Computation (B. Buchberger, G. E. Collins and R. Loos ed.), Springer-Verlag, Wien-New York, 1982.

[P1850] V. A. Puiseux, *Recherches sur les functions algébriques.* J. Math. **15** (1850), 365-480.

[SKK94] T. Sasaki, T. Kitamoto and F. Kako, *Error analysis of power series roots of multivariate algebraic equation.* Preprint of Univ. Tsukuba (March 1994), 30 pages (submitted).

[SS92] T. Sasaki and M. Sasaki, *A unified method for multivariate polynomial factorization.* Japan J. Indus. Appl. Math., **10** (1993), 21-39.

[SS96] K. Shiihara and T. Sasaki, *Analytic continuation and Riemann surface determination of algebraic functions by computer.* Japan J. Indus. Appl. Math., **13** (1996), 107-116.

[SSKS91] T. Sasaki, M. Suzuki, M. Kolář and M. sasaki, Approximate factorization of multivariate polynomials and absolute irreducibility testing. Japan J. Indus. Appl. Math., **8** (1991), 357-375.

[Wal78] R. J. Walker, *Algebraic curves.* Springer-Verlag, New York-Heidelberg-Berlin, 1978.

[Wal91] B. Wall, *Puiseux expansion and annotated bibliography.* Technical Report, RISC-Linz Series No. 91-46.0 (Oct. 1991).

[Wan79] P. S. Wang, *Parallel p-adic constructions in the univariate polynomial factoring algorithm.* Proc. 1979 MACSYMA Users' Conference, MIT Lab. Comp. Sci., MIT Boston, 1979, 310-318.