Computational constructions of W-graphs corresponding to Hecke algebras $H(q, n)$ for n up to 15†

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Abstract. We will construct all W-graphs corresponding to irreducible representations of Hecke algebras $H(q, n)$ for n up to 15 by a computer program using Lascoux-Schützenberger's method with auxiliary modifications. Our results heve been confirmed the validity by computational verifications.

1. Introduction.

V. Jones [Jo1] discovered a polynomial invariant in one variable which is an invariant of oriented knots and links, and later P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett, and A. Ocneanu [FYHLMO] gave generalizations of the Jones invariant in two variables.

Jones also defined in [Jo2] another two-variable invariant $X_L(q, \lambda)$ of an oriented link L given by the following formula

$$X_L(q, \lambda) = \left(-\frac{1 - \lambda q}{\sqrt{\lambda}(1 - q)} \right)^{n-1} (\sqrt{\lambda})^e tr(\pi(\alpha))$$

where α is any element of the braid group B_n with $\hat{\alpha} = L$, e being the exponent sum of α and π is the representation of B_n in the Hecke algebra $H(q, n)$ sending the standard generators of B_n to those of $H(q, n)$.

Ocneanu's trace $tr(g_i)$ for each generator g_i is defined by

$$tr(g_i) = \sum_Y W_Y(q, z) tr_Y(g_i)$$

where Y is a Young diagram associated with a partitions of n, tr_Y being the trace on the Hecke algebra obtained by evaluating the sum of the diagonal entries on the image of g_i in the matrix representation π_Y (see the precise definition in [Jo2]).

Two ways to compute $tr(g_i)$ are known. One is due to P. Hoefsmit [Ho] and H. Wenzl [We] which is not well adapted for computer calculations as it involves square roots of certain polynomials. The other is one proposed by A. Lascoux and M. Schützenberger [LS].

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Their method is combinatorial and uses W-graphs defined by Kazhdan and Lusztig [KL] for irreducible representations of the symmetric group S_n. It's explicit formula is expressed briefly in [Gy2,Gy3].

In this paper, we will construct all W-graphs corresponding to irreducible representations of Hecke algebras $H(q, n)$ for n up to 15 by using Lascoux-Schützenberger's method.

Our results obtained by computational constructions verify the validity of Lascoux-Schützenberger's method in the case when n is up to 13 (see Section 3).

After we submitted the first version of this paper, the referee informed us that Tim Maclarnan had already constructed a counterexample for Lascoux-Schützenberger's method in the case when $n = 14$ by a computational method around 1988. We also confirmed that the representation matrix obtained by this method does not satisfy the defining relations of $H(q, 14)$ when the matrix corresponds to one of the Young diagrams $\{5, 4, 3, 2\}$, $\{5, 4, 2, 2, 1\}$, and $\{4, 4, 3, 2, 1\}$.

But we can overcome the incompleteness of Lascoux-Schützenberger's method in the case when n is 14 and 15 with auxiliary modifications (see Section 3).

We would like to thank A. Gyoja, J. Murakami and H. Naruse for the useful discussions and advices. In particular, Gyoja informed us the precise definition of Lascoux-Schützenberger's method.

After finishing the first version of this work, the first author visited the Geometry center of the University of Minnesota, where he fixed some bugs in the computer program. He would like to express his thanks to former Managing Director A. Marden and also like to thank the Geometry center for its hospitality.

2. Lascoux-Schützenberger's method.

Let $\Lambda(n)$ be the set of partitions of a positive integer n, i.e.

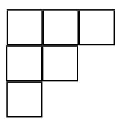
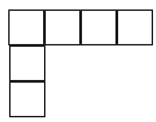
$$\Lambda(n) = \left\{ (\lambda_1, \lambda_2, \dots) \mid \sum_i \lambda_i = n, \lambda_i \in N, \lambda_i \geqq \lambda_{i+1} (i \in N) \right\}$$

and $|\Lambda(n)|$ be the number of elements in $\Lambda(n)$. For example, $\Lambda(6)$ is as follows:

$$\begin{aligned} \Lambda(6) = \{&(6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1), \\ &(2,2,2), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1)\} \\ |\Lambda(6)| =& 11. \end{aligned}$$

For each partition P of $\Lambda(6)$, a Young diagram Y can be defined diagrammatically as follows:

partition:$3 + 2 + 1 = 6$ partition:$4 + 1 + 1 = 6$

diagram:  diagram: 

where the length of the rows and columns in the diagram is made to be non-decreasing. We call Y *the Young diagram associated with* P, or also *a Young diagram associated with* $\Lambda(n)$.

Given a Young diagram Y, we assign each number in $\{1, 2, ..., n\}$ of each small box with the following conditions:

(1) the numbers in each row are assigned to be increasing,
(2) the numbers in each column are assigned to be increasing.

For example, a Young diagram $Y = \{3, 2, 1\}$ associated with $\Lambda(6)$ has the following 16 assignments:

$$\begin{array}{ccc}1&4&6\\2&5&\\3&&\end{array}\quad \begin{array}{ccc}1&3&6\\2&5&\\4&&\end{array}\quad \begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array}\quad \begin{array}{ccc}1&3&6\\2&4&\\5&&\end{array}\quad \begin{array}{ccc}1&2&6\\3&4&\\5&&\end{array}\quad \begin{array}{ccc}1&4&5\\2&6&\\3&&\end{array}$$

$$\begin{array}{ccc}1&3&5\\2&6&\\4&&\end{array}\quad \begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array}\quad \begin{array}{ccc}1&3&4\\2&6&\\5&&\end{array}\quad \begin{array}{ccc}1&2&4\\3&6&\\5&&\end{array}\quad \begin{array}{ccc}1&2&3\\4&6&\\5&&\end{array}\quad \begin{array}{ccc}1&3&5\\2&4&\\6&&\end{array}$$

$$\begin{array}{ccc}1&2&5\\3&4&\\6&&\end{array}\quad \begin{array}{ccc}1&3&4\\2&5&\\6&&\end{array}\quad \begin{array}{ccc}1&2&4\\3&5&\\6&&\end{array}\quad \begin{array}{ccc}1&2&3\\4&5&\\6&&\end{array}$$

Each such assignment is called a standard Young tableau. Hence $Y = \{3, 2, 1\}$ associated with $\Lambda(6)$ has 16 standard Young tableaux.

Given a standard Young tableau $x = \begin{array}{ccc}w_1&w_2&w_3\\w_4&w_5&\\w_6&&\end{array}$, we place numbers in x in a certain ordering (i.e., $w_6w_4w_5w_1w_2w_3$) yields an arrangement of n numbers. We call such an arrangement a *word*. For example, a standard Young tableau $x_1 = \begin{array}{ccc}1&4&6\\2&5&\\3&&\end{array}$ corresponds to a word 325146. Thus $Y = \{3, 2, 1\}$ associated with $\Lambda(6)$ yields 16 words as follows:

$$\begin{array}{lllll} x_1 = 325146, & x_2 = 425136, & x_3 = 435126, & x_4 = 524136, & x_5 = 534126, \\ x_6 = 326145, & x_7 = 426135, & x_8 = 436125, & x_9 = 526134, & x_{10} = 536124, \\ x_{11} = 546123, & x_{12} = 624135, & x_{13} = 634125, & x_{14} = 625134, & \\ x_{15} = 635124, & x_{16} = 645123. &&& \end{array}$$

Let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of words determined by a Young diagram Y associated with $\Lambda(n)$. Then a graph $G_{LS}(Y)$ with the vertex set X is defined by the following adjacency relations:

(1) If x and x' are two different vertices in $G_{LS}(Y)$ such that

$$\begin{aligned} x &= w_1 i w_2 j w_3, \\ x' &= w_1 j w_2 i w_3, \end{aligned}$$

where w_1, w_2 and w_3 are subwords in x which may be empty and w_2 does not contain the number within the range i and j, then x is adjacent to x'.

For example, $x_1 = 325146$ is adjacent to $x_2 = 425136$ in $G_{LS}(Y)$ but $x_2 = 425136$ and $x_{14} = 625134$ are not adjacent in $G_{LS}(Y)$ in this stage.

(2) Let x be a vertex in $G_{LS}(Y)$. For each triplet of 3 consecutive numbers i, $i+1$, $i+2$ $(1 \leqq i \leqq n-2)$, we define a vertex $x^{(i)}$ as follows:

$$x^{(i)} = \begin{cases} \text{is undefined,} & \text{if } x = w_1 i w_2 (i+1) w_3 (i+2) w_4 \\ & \text{or } x = w_1 (i+2) w_2 (i+1) w_3 i w_4 \\ w_1 (i+1) w_2 (i+2) w_3 i w_4, & \text{if } x = w_1 i w_2 (i+2) w_3 (i+1) w_4 \\ w_1 (i+2) w_2 i w_3 (i+1) w_4, & \text{if } x = w_1 (i+1) w_2 i w_3 (i+2) w_4 \\ w_1 i w_2 (i+2) w_3 (i+1) w_4, & \text{if } x = w_1 (i+1) w_2 (i+2) w_3 i w_4 \\ w_1 (i+1) w_2 i w_3 (i+2) w_4, & \text{if } x = w_1 (i+2) w_2 i w_3 (i+1) w_4, \end{cases}$$

where w_1, w_2, w_3 , and w_4 are subwords of x which may be empty.

For any pair of vertices x and $\bar{x}$ which is ajacent in $G_{LS}(Y)$, vertices $x^{(i)}$ and $\bar{x}^{(i)}$ defined above are also adjacent in $G_{LS}(Y)$.

For example, two vertices $x_2 = 425136$ and $x_4 = 524136$ are adjacent in $G_{LS}(Y)$ and so $x_2^{(2)} = x_1 = 325146$ and $x_4^{(2)} = x_5 = 534126$ are adjacent in $G_{LS}(Y)$.

This process is applied repeatedly so that there appears no more adjacent pairs.

The method mentioned above was proposed by Lascoux-Schützenberger in [LS] and [Gy2, Gy3]. We call it Lascoux-Schützenberger's method. For $Y = \{3, 2, 1\}$ associated with $\Lambda(6)$, $G_{LS}(Y)$ has the folloing adjacency matrix:

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}$$

3. Irreducible representations of Hecke algebras $H(q,n)$.

Let $H(q,n)$ be a C-algebra with a unit defined by the following relations:

$$H(q,n) = \Big\langle g_1, g_2, \ldots, g_{n-1} \Big| \quad \begin{array}{r} g_i^2 = (q-1)g_i + q, \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \\ g_i g_j = g_j g_i, \text{ if } |i-j| \geqq 2 \end{array} \Big\rangle$$

Then $H(q,n)$ is called a Hecke algebra of type A_{n-1} and each generator g_i is called a standard generator of it.

Let $\Lambda(n)$ be the set of partitions of a positive integer n, Y be a Young diagram associated with $\Lambda(n)$ and $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of words determined by Y.

For each element x of X, $I(x)$ is defined as a sub-collection $i \in I = \{1, 2, ..., n-1\}$ such that the row which contains the number i is higher than the one which contains the number $(i+1)$ in x as a standard Young tableau. We call $I(x)$ the I-set of x. If $x = x_{16}$ is a vertex in the preceding example $G_{LS}(\{3,2,1\})$, then $I(x) = \{3,5\}$.

Given a triplet $\{X, I, \mu\}$ where $\mu(x_l, x_m)$ is an arbitrary binary function, we can define square matrices T_j $(j = 1, 2, ..., n-1)$ of size s. The (l,m) elements of T_j are defined as follows:

$$T_j(l,m) = \begin{cases} -1, & l = m \text{ and } j \in I(x_l) \\ q, & l = m \text{ and } j \notin I(x_l) \\ \sqrt{q}, & l \neq m, j \in I(x_l), j \notin I(x_m), \\ & \mu(x_l, x_m) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then a triplet $\{X, I, \mu\}$ is called a W-graph corresponding to Y, if all those matrices satisfy the defining relations of Hecke algebras $H(q,n)$ under representation π_Y with $\pi_Y(g_j) \equiv T_j (j = 1, 2, ..., n-1)$ (see [Gy1,KL]). We denote $\{X, I, \mu\}$ by $G_W(Y)$, if it is a W-graph. Lascoux Schützenberger conjectured in [LS] and [Gy2,Gy3] that any triplet $\{X, I, \mu\}$ is a W-graph, where μ is the binary function defined by the adjacency relations of $G_{LS}(Y)$.

From now on, we will demonstrate that this conjecture is true for n up to 13, but it is false for $n = 14$ and 15. And there need some modifications to Lascoux-Schützenberger method.

At first, we made computer software to construct all $G_{LS}(Y)$ and all I-sets of Young diagrams Y associated with $\Lambda(n)$, and executed computational constructions.

We performed direct matrix calculations to check whether the resulting matrices satisfy the defining relations of Hecke algebras $H(q,n)$ and found that 3 of 135 representations for the case of $n = 14$ and 21 of 176 representations for the case of $n = 15$ do not satisfy the necessary relations. In these cases, they do not satisy two of the three types of defining relations for Hecke algebras $H(q,n)$, namely the conjugacy and commutation relations.

Very recently, Naruse made a W-graph associated with the Young diagram $\{4, 4, 3, 2, 1\}$ using Kazhdan-Lusztig polynomials [Na] by also computational construction. We compare his results with ours and found that there are 68 edges such that Lascoux-Schützenberger's method faild to detect. These edges are generated from the following 8 edges with step 2 of Lascoux-Schützenberger's method.

87C36B25AE149D $\leftrightarrow$ C8A36E25BD1479, 87C36B25AE149D $\leftrightarrow$ C8E6AB279D1345,
C4837B26AE159D $\leftrightarrow$ C8E4AB267D1359, C7B36A259E148D $\leftrightarrow$ CAE6BD27891345,
76B5AE249D138C $\leftrightarrow$ EAB67C248D1359, B6A59E248D137C $\leftrightarrow$ EAB68C249D1357,
D6A59E248C137B $\leftrightarrow$ DAE68C249B1357, A6E59D248C137B $\leftrightarrow$ EAC68D249B1357.

Here we expressed the number 10 as "A" and 11 as "B", ...

This result suggests that there are some edges which Lascoux-Schützenberger's method failed to find. Although we can't make rigid algorithm to find them, we make following heuristic method:

(1) Calculate ajacency matrix and $I(x)$ for each word x by Lascoux-Schützenberger's method.
(2) Calculate T_i, $(i = 1, \ldots, n-1)$.
(3) Check the commutation relations:

$$C_{i,j} = T_i T_j - T_j T_i, \quad (i = 1, \ldots, n-3,\ j = i+2, \ldots, n-1).$$

For each non vanishing commutation relations, pick up non zero element of $C_{i,j}$.

For example, if (l, m) element of $C_{i,j}$ is non zero, then the candidates for auxiliary edge are any pair of $x_l \leftrightarrow x_k$ such that (l, k) element of T_i or T_j is nonzero or $x_k \leftrightarrow x_m$ such that (k, m) element of T_i or T_j is nonzero. Adding one of these candidates and all the edges generated from step 2 of Lascoux-Schützenberger's method. Then recalculate T_i, $(i = 1, \ldots, n-1)$ and check the commutation relations. If number of nonzero elements of resulting commutation relations is less than the original, then we modify Lascoux-Schützenberger's method to add this edge.
(4) These procedure is performed until all the commutation relations vanish.

All the auxiliary edges obtained are listed in Appendix.

After finishing the first stage of Lascoux-Schützenberger's method, we add the auxiliary edges mentioned above and finally get all the correct representation matrices. That is, all the resulting matrices obtained by Lascoux-Schützenberger's method with the modification mentioned above satisfy the defining relations of Hecke algebras $H(q, n)$.

Our current program can get all $G_W(Y)$ corresponding to Young diagrams Y associated with $\Lambda(n)$ and all T_j $(i = 1, 2, ..., n-1)$ for n up to 15 in about one week on the computer Sun Microsystems SPARCserver 1000 with 160 Mega bytes of main memory. For $n = 15$, it takes about 137 hours of cpu time to calculations and needs 87 Mega bytes of main memory.

It is well known that the representation which is given by a W-graph corresponding to a Young diagram is irreducible (see [Gy1, KL]). Hence the matrices obtained by our computational constructions give the irreducible representation of Hecke algebras $H(q, n)$.

4. Listings of W-graphs corresponding to $H(q, n)$ for n up to 15.

We give a sample list of some W-graphs corresponding to $H(q, n)$, and we provide comments so as to make these lists more understandable. W-graphs corresponding to $H(q, n)$ for n up to 15 are too big to publish. So, we hope to have our computer program available on the ftp server at the Geometry Center so that readers in this field of knot theory, representation theory, and etc. can use them.

Young diagram $Y = \{2,2,2,2\}$ in $\Lambda(8)$

x_1	[1 2 3 5 6 7]	$\{x_2, x_6, x_{12}\}$,	x_2	[1 2 4 6 7]	$\{x_1, x_3, x_5, x_{14}\}$,
x_3	[1 3 4 6 7]	$\{x_2, x_4, x_6, x_{13}\}$,	x_4	[2 3 4 6 7]	$\{x_3, x_7, x_{14}\}$,
x_5	[1 2 4 5 7]	$\{x_2, x_6, x_9, x_{10}\}$,	x_6	[1 3 5 7]	$\{x_1, x_3, x_5, x_7, x_8, x_{11}\}$,
x_7	[2 3 5 7]	$\{x_4, x_6, x_9, x_{12}\}$,	x_8	[1 3 4 5 7]	$\{x_6, x_9, x_{13}\}$,
x_9	[2 4 5 7]	$\{x_5, x_7, x_8, x_{14}\}$,	x_{10}	[1 2 4 5 6]	$\{x_5, x_{11}, x_{14}\}$,
x_{11}	[1 3 5 6]	$\{x_6, x_{10}, x_{12}, x_{13}\}$,	x_{12}	[2 3 5 6]	$\{x_1, x_7, x_{11}, x_{14}\}$,
x_{13}	[1 3 4 6]	$\{x_3, x_8, x_{11}, x_{14}\}$,	x_{14}	[2 4 6]	$\{x_2, x_4, x_9, x_{10}, x_{12}, x_{13}\}$

This list gives the W-graph $G_W(Y)$ corresponding to a Young diagram $Y = \{2,2,2,2\}$ associated with $\Lambda(8)$. There are 14 vertices $x_1, x_2, \ldots, x_{14}$ and x_1 has I-set [1 2 3 5 6 7], x_2 I-set [1 2 4 6 7], ..., and x_{14} I-set [2 4 6]. The next list means that x_1 is adjacent to x_2, x_6 and x_{12}, x_2 adjacent to x_1, x_3, x_5 and x_{14},..., and x_{14} is adjacent to x_2, x_4, x_9, x_{10}, x_{12}, x_{13}.

5. Final remarks.

The first author and J. Murakami established the 3-parallel version of polynomial invariants of closed 3 and 4 braids associated with certain subspaces of representation matrices of the irreducible representation of $H(q,n)$ for n of 9 and 12 in [OM].

We are now calculating 3-parallel version polynomial invariants of closed 5 braids using irreducible representations of $H(q,15)$. In the forthcomming paper, we will show the results.

It remains unknown an effective algorithm to construct irreducible representation of $H(q,n)$ for large n.

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Appendix.

In this appendix, we give the list of representations which Lascoux-Schützenberger's algorithm failed, and table of all the auxiliary edges to modify this algorithm for $n = 14$ and 15. Because Young diagram associated with partition $\{4, 4, 3, 2, 1\}$ is adjoint to diagram with $\{5, 4, 3, 2\}$, so the auxiliary edges necessary for diagram $\{4, 4, 3, 2, 1\}$ are easily obtained from that of diagram $\{5, 4, 3, 2\}$. We omit the table for diagram $\{4, 4, 3, 2, 1\}$, etc. We also omit edges which can be generated by stage 2 of Lascoux-Schützenberger's method.

braid	matrix size	Young diagram	Number of auxiliary edges
14	48048	{5,4,3,2}	68
14	68640	{5,4,2,2,1}	50
14	48048	{4,4,3,2,1}	68
15	30030	{6,5,4}	8
15	128700	{6,5,3,1}	68
15	100100	{6,5,2,2}	48
15	175175	{6,4,3,2}	322
15	243243	{6,4,2,2,1}	250
15	54054	{5,5,4,1}	48
15	96525	{5,5,3,2}	232
15	125125	{5,5,2,2,1}	110
15	81081	{5,4,4,2}	80
15	75075	{5,4,3,3}	100
15	292864	{5,4,3,2,1}	1720
15	125125	{5,4,2,2,2}	110
15	243243	{5,4,2,2,1,1}	250
15	75075	{4,4,4,2,1}	100
15	81081	{4,4,3,3,1}	80
15	96525	{4,4,3,2,2}	232
15	175175	{4,4,3,2,1,1}	322
15	100100	{4,4,2,2,2,1}	48
15	54054	{4,3,3,3,2}	48
15	128700	{4,3,3,2,2,1}	68
15	30030	{3,3,3,3,2,1}	8

Table 2. Representations which Lascoux-Schützenberger's algorithm failed

Table 3. Table of auxiliary edges

Young diagram $Y = \{5, 4, 3, 2\}$ in $\Lambda(14)$

7B59D348C126AE $\leftrightarrow$ BC78D349E1256A, 7D59C348B126AE $\leftrightarrow$ CD78E349A1256B,
9D58C347B126AE $\leftrightarrow$ CD89E34AB12567, 7B59C348E126AD $\leftrightarrow$ BC78E349A1256D,
5948D37BE126AC $\leftrightarrow$ DE89A456B1237C, 5D49B37AE1268C $\leftrightarrow$ DE9AB456C12378,
9D57B36AE1248C $\leftrightarrow$ DE9AB56781234C, 9D7BE456A1238C $\leftrightarrow$ DE9AB456C12378.

Young diagram $Y = \{5, 4, 2, 2, 1\}$ in $\Lambda(14)$

B6A59248D137CE $\leftrightarrow$ B9D5A24CE13678, 87C6B35AE1249D $\leftrightarrow$ CBE78359A1246D.

Young diagram $Y = \{6, 5, 4\}$ in $\Lambda(15)$

47AD369CE1258BF $\leftrightarrow$ ACDE4678F12359B, 58BE47ADF12369C $\leftrightarrow$ BDEF5789A12346C.

Young diagram $Y = \{6, 5, 3, 1\}$ in $\Lambda(15)$

847B26ADF1359CE $\leftrightarrow$ B7DF289AC13456E, 76AD459CF1238BE $\leftrightarrow$ D79F45ABC12368E,
B7AE4569D1238CF $\leftrightarrow$ EABC456DF123789, C58B347AE1269DF $\leftrightarrow$ C8AE34BDF125679,
958D347CF126ABE $\leftrightarrow$ D89F34ABC12567E, A59D348CF1267BE $\leftrightarrow$ D9AF34BCE125678.

Young diagram $Y = \{6, 5, 2, 2\}$ in $\Lambda(15)$

7D6A459CF1238BE $\leftrightarrow$ DF7945ABC12368E, 8C5B347AE1269DF $\leftrightarrow$ CE8A34BDF125679,
9D58347CF126ABE $\leftrightarrow$ DF8934ABC12567E, AD59348CF1267BE $\leftrightarrow$ DF9A34BCE125678.

Young diagram $Y = \{6, 4, 3, 2\}$ in $\Lambda(15)$

5948C37BE126ADF $\leftrightarrow$ CE89A456B1237DF, 8C47B36AE1259DF $\leftrightarrow$ CE8AB46791235DF,
5C48B37AE1269DF $\leftrightarrow$ CE8AB456D12379F, 7B6AE459D1238CF $\leftrightarrow$ BD79E45AF12368C,
7B6AD459F1238CE $\leftrightarrow$ BD79F45AC12368E, 7E6AD459C1238BF $\leftrightarrow$ DE79F45AB12368C,
AE69D348C1257BF $\leftrightarrow$ DE9AF34BC125678, 9D58C347B126AEF $\leftrightarrow$ CD89E34AB12567F,
8C7BE346A1259DF $\leftrightarrow$ CE8AB346D12579F, 9E58D347C126ABF $\leftrightarrow$ DE89F34AB12567C,
6A59D48CF1237BE $\leftrightarrow$ DF9AB567C12348E, 9D58C47BF1236AE $\leftrightarrow$ DF9BC578A12346E,
6D59C48BF1237AE $\leftrightarrow$ DF9BC567E12348A, 5A49D38CF1267BE $\leftrightarrow$ DF9AB456C12378E,
5D49C38BF1267AE $\leftrightarrow$ DF9BC456E12378A, 9D8CF346B1257AE $\leftrightarrow$ DF9BC346E12578A,
6A59D348C127BEF $\leftrightarrow$ AC68D349E1257BF, 6A59C348E127BDF $\leftrightarrow$ AC68E349B1257DF,
6D59C348B127AEF $\leftrightarrow$ CD68E349A1257BF, 6E59D348C127ABF $\leftrightarrow$ DE68F349A1257BC.

Young diagram $Y = \{6, 4, 2, 2, 1\}$ in $\Lambda(15)$

B6A49258D137CEF $\leftrightarrow$ B9D4A25CE13678F, 87C6B45AE1239DF $\leftrightarrow$ CBE78459A1236DF,
C7B5A269E1348DF $\leftrightarrow$ CAE5B26DF134789, 98D7C34BF1256AE $\leftrightarrow$ DCF8934AB12567E.

Young diagram $Y = \{5, 5, 4, 1\}$ in $\Lambda(15)$

A369D258CF147BE ↔ A6CDF289BE13457, B47AF369CE1258D ↔ FABCD4678E12359,
837BE26ADF1459C ↔ B7DEF289AC13456, 968CF257BE134AD ↔ F9BCD5678E1234A.

Young diagram $Y = \{5, 5, 3, 2\}$ in $\Lambda(15)$

5C48B37AEF1269D ↔ CE8AB456DF12379, 7B6AD459CF1238E ↔ BD79F45ACE12368,
7E6AD459CF1238B ↔ DE79F45ABC12368, 5A49D368CF127BE ↔ DF9AB456CE12378,
6A59D248CF137BE ↔ DF9AB567CE12348, 6D59C248BF137AE ↔ DF9BC5678E1234A,
9D58C347BF126AE ↔ CD89E34ABF12567, 8C7BE346AF1259D ↔ CE8AB346DF12579,
7B46A359DF128CE ↔ BD79F34ACE12568, 9E58D347CF126AB ↔ DE89F34ABC12567,
8C7BF256AE1349D ↔ CF8AB256DE13479, 6A59D248CF137BE ↔ AC68D249EF1357B,
6D59C248BF137AE ↔ CD68E249AF1357B, 9D58C247BF136AE ↔ DF9BC578AE12346.

Young diagram $Y = \{5, 5, 2, 2, 1\}$ in $\Lambda(15)$

B6A49258DF137CE ↔ B9D4A25CEF13678, D6A49258CF137BE ↔ D9F4A25BCE13678,
98D7C246BF135AE ↔ DCF8926ABE13457, 98D5C247BF136AE ↔ DCF8924ABE13567.

Young diagram $Y = \{5, 4, 4, 2\}$ in $\Lambda(15)$

7B36AE259D148CF ↔ BD79EF356A1248C, 5948CF37BE126AD ↔ CE89AF456B1237D,
8C47BF36AE1259D ↔ CE8ABF46791235D, 5C48BF37AE1269D ↔ CE8ABF456D12379,
7E36AD259C148BF ↔ DE79AF356B1248C, AE369D258C147BF ↔ DE9ABF356C12478,
6A59DF248C137BE ↔ DF9ABC56781234E, 6D59CF248B137AE ↔ DF9BCE56781234A,
9D58CF347B126AE ↔ CD89EF34AB12567, 6A59CF248E137BD ↔ AC68EF249B1357D,
6D59CF248B137AE ↔ CD68EF249A1357B, 9D38CF257B146AE ↔ DF9BCE35781246A.

Young diagram $Y = \{5, 4, 3, 3\}$ in $\Lambda(15)$

7BE36A259D148CF ↔ BDE79A356F1248C, 7BD36A259F148CE ↔ BDF79A356C1248E,
59F48C37BE126AD ↔ CEF89A456B1237D, 8CF47B36AE1259D ↔ CEF8AB46791235D,
5CF48B37AE1269D ↔ CEF8AB456D12379, 7AE36D259C148BF ↔ DEF79A356B1248C,
59D48C27BF136AE ↔ 9DF5BC278A1346E, 8CF7BE346A1259D ↔ CEF8AB346D12579,
6AE59D248C137BF ↔ ACE68D249F1357B, 6AE39D258C147BF ↔ DEF9AB356C12478.

Young diagram $Y = \{5,4,3,2,1\}$ in $\Lambda(15)$

C7B36A259E148DF $\leftrightarrow$ CAE3BD267F14589, A5948D37CF126BE $\leftrightarrow$ D9F4AB357C1268E,
98D67C25BF134AE $\leftrightarrow$ D9F6BC278A1345E, 98D57C46BF123AE $\leftrightarrow$ DCF89A456B1237E,
D5948C37BF126AE $\leftrightarrow$ D9F4BC357E1268A, 76B5AE249D138CF $\leftrightarrow$ B7D59E24AF1368C,
B6A59E248D137CF $\leftrightarrow$ B9D5AE24CF13678, 76B5AD249F138CE $\leftrightarrow$ B7D59F24AC1368E,
B6A59F248D137CE $\leftrightarrow$ B9D5AF24CE13678, 87C6BF34AE1259D $\leftrightarrow$ CBE78F349A1256D,
87C36B25AE149DF $\leftrightarrow$ C8A36E25BD1479F, D8C7BF456A1239E $\leftrightarrow$ DCF8AB456E12379,
C4837B26AE159DF $\leftrightarrow$ C8E4AB267D1359F, 87C36B25AE149DF $\leftrightarrow$ C8E6AB279D1345F,
C7B36A259E148DF $\leftrightarrow$ CAE6BD27891345F, 76B5AE349D128CF $\leftrightarrow$ EAB67C348D1259F,
76B5AE349D128CF $\leftrightarrow$ BAD67E348F1259C, C4B37A269E158DF $\leftrightarrow$ CAE4BD267F13589,
76E5AD349C128BF $\leftrightarrow$ EAC67D348F1259B, A9E58D347C126BF $\leftrightarrow$ EAC89D34BF12567,
B6A59E348D127CF $\leftrightarrow$ BAE68D349C1257F, B6A59E348D127CF $\leftrightarrow$ EAB68C349D1257F,
A6E59D348C127BF $\leftrightarrow$ EAC68D349B1257F, B6A59E348D127CF $\leftrightarrow$ BAD68E349F1257C,
76E5AD249C138BF $\leftrightarrow$ D7E59F24AB1368C, E6A59D348C127BF $\leftrightarrow$ EAC68D349F1257B,
B6A59D348F127CE $\leftrightarrow$ BAD68F349C1257E, A9E58D347C126BF $\leftrightarrow$ D9E8AF34BC12567,
D6E59C348B127AF $\leftrightarrow$ DCE68F349A1257B, A6E59D248C137BF $\leftrightarrow$ D9E5AF24BC13678,
C7B6AF349E1258D $\leftrightarrow$ CBF79E34AD12568, D8C37B26AF1459E $\leftrightarrow$ DBF3CE27891456A,
87C6BF34AE1259D $\leftrightarrow$ FBC78D349E1256A, C7B6AF349E1258D $\leftrightarrow$ FBC79D34AE12568,
87F6BE34AD1259C $\leftrightarrow$ FBD78E349A1256C, B7F6AE349D1258C $\leftrightarrow$ FBD79E34AC12568,
D9E58C347B126AF $\leftrightarrow$ DCE89F34AB12567, E9D58C347B126AF $\leftrightarrow$ ECD89F34AB12567,
A5948D37CF126BE $\leftrightarrow$ D9F8AB456C1237E, D5948C37BF126AE $\leftrightarrow$ DCF89A456B1237E,
98D47C36BF125AE $\leftrightarrow$ DCF89A467B1235E, D8C47B36AF1259E $\leftrightarrow$ DCF8AB46791235E,
D5C48B37AF1269E $\leftrightarrow$ DCF8AB456E12379, D7C36B25AF1489E $\leftrightarrow$ DBF6CE27891345A,
A9E46D358C127BF $\leftrightarrow$ EAC46D358F1279B, E6D59C348B127AF $\leftrightarrow$ ECD68F349A1257B,
76C5BF34AE1289D $\leftrightarrow$ FBC67D348E1259A, 65A49D28CF137BE $\leftrightarrow$ D9F5AB267C1348E,
65D49C28BF137AE $\leftrightarrow$ D9F5BC267E1348A, 98D47C26BF135AE $\leftrightarrow$ D9F7BC28AE13456,
D8C47B26AF1359E $\leftrightarrow$ DBF7CE289A13456, A5948D27CF136BE $\leftrightarrow$ A9D5CF278B1346E,
98D57C46BF123AE $\leftrightarrow$ D9B57F46CE1238A, A5948D27CF136BE $\leftrightarrow$ D9F5AB278C1346E,
D5C48B27AF1369E $\leftrightarrow$ DBF5CE27891346A, 95D48C27BF136AE $\leftrightarrow$ D9F5BC278A1346E,
98D4CF257B136AE $\leftrightarrow$ D9F4BC257E1368A, D5948C27BF136AE $\leftrightarrow$ D9F5BC278E1346A.

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