# On Cancellation Error in Newton's Method for Power Series Roots of Multivariate Polynomial *

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**Abstract**

Let $F(x, u_1, \ldots, u_\ell)$ be a square-free polynomial which is monic w.r.t. $x$ and let $(s_1, \ldots, s_\ell) \in \mathbf{C}^\ell$. If $F(x, s_1, \ldots, s_\ell)$ is square-free then the roots of $F$ w.r.t. $x$ can be expanded into integral power series in $u_1 - s_1, \ldots, u_\ell - s_\ell$, and the power series roots can be calculated by well-known Newton's method. We use the floating-point number arithmetic to calculate the numerical coefficients, and this paper investigates the numerical errors contained in the roots computed. We first express the power series root in terms of sub-polynomials of $F(x, s_1 + v_1, \ldots, s_\ell + v_\ell)$, with $(v_1, \ldots, v_\ell)$ a set of new variables, and clarify various properties of the power series roots. Then, we investigate the cancellation errors when the expansion point $(s_1, \ldots, s_\ell)$ is close to a "singular point" and far from the origin, respectively. We find that, although the cancellation errors are not large usually, they become extremely large in some cases. In fact, we will encounter very large errors of magnitude $O(C^k)$, $C > 1$, in the $k$-th power terms. We clarify in which case the errors become large and what magnitudes the errors are. Furthermore, we present an error-safe expansion method near the singular point.

*Key words*: approximate algebra, approximate algebraic computation, cancellation error, error analysis, multivariate polynomial equation, Newton's method, power series root.

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# 1 Introduction

Let $F(x, u_1, \ldots, u_\ell)$ be a monic square-free polynomial in variables $x$ and $u_1, \ldots, u_\ell$, and let $\chi(u_1, \ldots, u_\ell)$ be a root w.r.t. $x$, of $F$. The root is usually an algebraic function in $u_1, \ldots, u_\ell$. It is well-known that if $F(x, s_1, \ldots, s_\ell)$, with $(s_1, \ldots, s_\ell) \in \mathbf{C}^\ell$, has no multiple root then the root $\chi(u_1, \ldots, u_\ell)$ can be expanded into integral power series in $u_1 - s_1, \ldots, u_\ell - s_\ell$. The power series roots can be calculated directly from $F(x, u_1, \ldots, u_\ell)$ by well-known Newton's method, see [KT78] or [GCL92] for example.

However, rigorous calculation of the power series root is very time-consuming because Newton's method requires an algebraic number $\alpha$ which is defined by $F(\alpha, s_1, \ldots, s_\ell) = 0$. This is a bottleneck for wider application of the power series roots to practical problems. This bottleneck can be bypassed by computing the coefficients of power series numerically hence approximately. We call a power series with approximate coefficients *approximate power series*. Approximate power series roots can be computed very efficiently by Newton's method. Although quadratically convergent Newton's method is popular, we consider in this paper only linearly convergent Newton's method which is simple and very efficient, too.

In handling polynomials with floating-point number coefficients, we must be very careful for errors in the numerical coefficients. Especially, we must investigate the *cancellation errors* caused by the cancellation of dominant terms; see **2.2** of this paper for the cancellation errors. As for numerical errors in algebraic computation with floating-point number arithmetic, Ku and Adler [KA69] and Sasaki and Sato [SS98] investigated various algorithms for symbolic determinant computation, Sasaki and Sasaki [SS89] analyzed the Euclidean algorithm for univariate polynomials having close roots, and Sasaki and Yamaguchi [SY98] studied the multivariate Hensel construction. All of these studies revealed that extremely large errors are often caused by the term cancellation. To authors' knowledge, however, no error analysis has been made for Newton's method. In this paper, we will clarify the appearance of large cancellation errors and determine the magnitudes of them in several characteristic cases.

In **2**, we review Newton's method and explain the cancellation errors. Newton's method breaks down if the expansion point is chosen at a "singular point". Hence, we explain the expansion method of the root at the singular point, which is indispensable for analyzing the roots expanded near a singular point. In **3**, we analyze Newton's method theoretically. We show that, if we divide the polynomial $F(x, s_1 + v_1, \ldots, s_\ell + v_\ell)$ into sub-polynomials in a suitable way, the $k$-th power term of the root can be expressed by a sum of products of these sub-polynomials. By this, we clarify various properties of the power series roots. In **4**, we investigate the cases in which the expansion point is close to a singular point and far from the origin, respectively. We estimate the magnitudes of dominant terms which appear in the computation of power series root, and estimate the magnitudes of cancellation errors. We find that Newton's method is safe against the cancellation error in many cases, while it causes extremely large cancellation errors in some cases. In **5**, we propose an error-safe expansion method near the singular point, investigate a case in which the expansion point is far from the origin but close to a singular point, and consider the accumulation of rounding errors very roughly.

We think that, although this paper gives only the order estimation of cancellation errors, it provides a prototype of error analysis of approximate algebraic algorithms.

# 2 Power series roots of algebraic equation

In this section, we first review Newton's method for calculating the power series roots. Then, we show by a simple example that, if we employ the floating-point number arithmetic, we sometimes encounter extremely large numerical errors in the coefficients. Such large errors happen if the expansion point is chosen near a special point which we call a *singular point*, and Newton's method breaks down at the singular point. Therefore, we describe a method for expanding the root at a singular point and clarify some properties of the roots expanded.

We use symbols $x$ and $u_1, \ldots, u_\ell$ as the main variable and sub-variables, respectively. By $\deg(P)$ we denote the degree w.r.t. $x$, of polynomial $P$. Let $T = cu_1^{e_1} \cdots u_\ell^{e_\ell}$, with $c$ a number. By $\mathrm{tdeg}(T)$ we denote the *total-degree* of $T$: $\mathrm{tdeg}(T) = e_1 + \cdots + e_\ell$. By $\mathrm{tdeg}(P)$ and $\mathrm{ord}(P)$, with $P$ a polynomial in $u_1, \ldots, u_\ell$, we denote the total-degree and the *order* of $P$, respectively, i.e., the minimum and the maximum, respectively, of total-degrees of the terms of $P$. Let $R = N/D$ be a rational function, where $N$ and $D$ are polynomials in $u_1, \ldots, u_\ell$. The order of $R$ is defined as $\mathrm{ord}(R) = \mathrm{ord}(N) - \mathrm{ord}(D)$. If $N$ and $D$ are homogeneous then $R$ is called *homogeneous*. By $\mathbf{C}[u_1, \ldots, u_\ell]$, $\mathbf{C}\{u_1, \ldots, u_\ell\}$ and $\mathbf{C}(u_1, \ldots, u_\ell)$, with $\mathbf{C}$ the field of complex numbers, we denote the polynomial ring, the formal power series ring and the rational function field, respectively, over $\mathbf{C}$ in $u_1, \ldots, u_\ell$. By $\mathbf{C}\{(u_1, \ldots, u_\ell)\}$ we denote the ring of series of homogeneous rational functions of nonnegative orders. We define the *norm* of polynomial $P$, to be expressed as $\|P\|$, by the maximum of absolute values of the numerical coefficients of $P$. By $\gcd(A, B)$ and $\langle p_1, \ldots, p_l \rangle$ we denote the greatest common divisor of $A, B$ and an ideal generated by $p_1, \ldots, p_l$, respectively. By $[G(x, v_1, \ldots, v_\ell)]_k^l$, with $G$ a polynomial in $x$ and $v_1, \ldots, v_\ell$ and $k \le l$, we denote the sum of all the terms of total-degree $i$, $k \le i \le l$, w.r.t. $v_1, \ldots, v_\ell$, of $G$. We express $[G(x, v_1, \ldots, v_\ell)]_k^k$ simply as $[G(x, v_1, \ldots, v_\ell)]_k$.

Below, we abbreviate the lists of variables and numbers $(u_1, \ldots, u_\ell)$ and $(s_1, \ldots, s_\ell)$, etc., to $(u)$ and $(s)$, etc., respectively.

## 2.1 Power series roots and Newton's method

Let $F(x, u)$ be a polynomial in $\mathbf{C}[x, u]$, expressed as

$$F(x, u) = f_n(u)x^n + \cdots + f_0(u)x^0. \tag{2.1}$$

$F(x, u)$ is *monic* if $f_n = 1$. Any non-monic polynomial $F(x, u)$ can be converted to a monic polynomial $\tilde{F}(x, u)$ by the transformation

$$F(x, u) \mapsto f_n^{n-1}(u)\, F(x/f_n,\, u) \overset{\text{def}}{=} \tilde{F}(x, u). \tag{2.2}$$

Let $\tilde{\chi}(u)$ be a root of $\tilde{F}(x, u)$ w.r.t. $x$, then $\chi(u) = \tilde{\chi}(u)/f_n(u)$ is a root of $F(x, u)$. Therefore, we assume without loss of generality that $F(x, u)$ is monic.

Let $(s_1, \ldots, s_\ell) \in \mathbf{C}^\ell$ and let $\alpha_1, \ldots, \alpha_n$ be the roots of $F(x, s)$, where we assume that $\alpha \overset{\text{def}}{=} \alpha_1$ is a simple root:

$$\begin{cases} F(x, s) = (x - \alpha_1) \cdots (x - \alpha_n), \\ \alpha_1, \ldots, \alpha_n \in \mathbf{C}, \quad \alpha_1 \ne \alpha_j \;\; (\forall j \ne 1). \end{cases} \tag{2.3}$$

We introduce new variables $v_1, \ldots, v_\ell$ as

$$v_1 \stackrel{\text{def}}{=} u_1 - s_1, \ \cdots \ , \ v_\ell \stackrel{\text{def}}{=} u_\ell - s_\ell. \tag{2.4}$$

We can expand a root $\chi(u)$ of $F(x, u)$, w.r.t. $x$, into a power series in $v_1, \ldots, v_\ell$. The expansion method is as follows.

First, we define the ideal $I$ as

$$I = \langle u_1 - s_1, \ \ldots, \ u_\ell - s_\ell \rangle = \langle v_1, \ldots, v_\ell \rangle. \tag{2.5}$$

Let $\chi^{(k)}(v)$ be a polynomial of total-degree $k$ w.r.t. $v_1, \ldots, v_\ell$, satisfying

$$F(\chi^{(k)}(v), s + v) \ \equiv \ 0 \quad (\mathrm{mod}\ I^{k+1}). \tag{2.6}$$

We call $\chi^{(k)}(v)$ the *$k$-th order approximation* of a power series root of $F(x, u)$. Obviously, $\chi^{(0)} = \alpha$ is the zeroth order approximation of the root. We can calculate $\chi^{(k)}(v)$ $(k \geq 1)$ iteratively by the formula [KT78,GCL92]

$$\chi^{(k)}(v) \ \equiv \ \chi^{(k-1)}(v) - \frac{F(\chi^{(k-1)}(v), s + v)}{F'(\alpha, s)} \quad (\mathrm{mod} \qquad k = 1, 2, \ldots, \tag{2.7}$$

where $'$ denotes the derivative w.r.t. $x$: $F' = \partial F / \partial x$. With (2.7), we obtain $\chi^{(k)}(v)$ in the following form:

$$\chi^{(k)}(v) \ = \ \alpha + y_1(v) + \cdots + y_k(v), \quad \mathrm{ord}(y_j) = j \ \ (j = 1, \ldots, k). \tag{2.8}$$

The approximate root $\chi^{(k)}(v)$ is nothing but the Taylor series of the root $\chi(s+v)$, expanded in variables $v_1, \ldots, v_\ell$ to the order $k$.

Formula (2.7) is Newton's iteration formula of linear convergence. Although (2.7) can be generalized easily to a formula of quadratic convergence [KT78], we consider only (2.7) in this paper because it is the simplest yet it is efficient.

## 2.2 Cancellation errors and assumptions

Formula (2.7) requires computation with a number $\alpha$, a root of $F(x, s)$. Representing $\alpha$ by a fixed-precision floating-point number, we encounter numerical errors in the coefficients of power series $\chi^{(k)}(v)$. We show how serious the errors are by an example.

**Example 1** Occurrence of extremely large numerical errors. [1]

$$\begin{aligned} F(x, u) \ = \ & x^6 - 3(u-1)x^4 - 2ux^3 + 3(u^2 - 2u + 1)x^2 \\ & - 6(u^2 - u)x - u^3 + 4u^2 - 3u + 1. \end{aligned} \tag{2.9}$$

We choose the expansion point at $s = 0.517$. Formula (2.7), with $\alpha$ chosen to be one of the close roots of $F(x, 0.517)$, gives us $\chi^{(4)}(v)$ as follows.

$$\begin{aligned} \chi^{(4)}(v) \ = \ & (-\underline{0.401297867627788}0536 - \underline{0.000086281544616}70816683\, i) \\ + \ & (-\underline{0.25873492432}55296906 - \underline{1.16758511345}8430428\, i)\, v \\ + \ & (\ \ \underline{0.1668181}426979336807 - \underline{0.0834450}7519359525892\, i)\, v^2 \\ + \ & (-\underline{0.179}3895553538013117 - \underline{0.695}8222465143548943\, i)\, v^3 \\ + \ & (\ \ 2.002104809488503015 \quad - 2.156471842392122295\, i)\, v^4. \end{aligned}$$

[1] Authors thank Mr. Shiihara for showing us this example.

Here, the underlined figures are correct while the others are wrong. Hence, in this example, the accuracy decreases very rapidly as the computation proceeds. In fact, $v^4$-term of $\chi^{(4)}$ is wholly erroneous. $\square$

The very large errors in Example 1 are caused by the cancellation of almost the same numbers. Let $T_1$ and $T_2$ be the same terms with almost the same coefficients:

$$\begin{cases} T_1 = (c_0 + c_1' + \epsilon_1) u_1^{e_1} \cdots u_\ell^{e_\ell}, & |c_1'| \ll |c_0|, \\ T_2 = (c_0 + c_2' + \epsilon_2) u_1^{e_1} \cdots u_\ell^{e_\ell}, & |c_2'| \ll |c_0|, \end{cases}$$

where $\epsilon_1$ and $\epsilon_2$ are small unknown errors and the leading figures of $c_1'$ and $c_2'$ are different. Then, the relative error of $T_1 - T_2$ becomes very large:

$$T_1 - T_2 = (c_1' - c_2' + \epsilon_1 - \epsilon_2) u_1^{e_1} \cdots u_\ell^{e_\ell}, \quad \frac{|\epsilon_1 - \epsilon_2|}{|c_1' - c_2'|} \gg \max\left\{ \frac{|\epsilon_1|}{|c_0|}, \frac{|\epsilon_2|}{|c_0|} \right\}.$$

This kind of error is *cancellation error* which may become quite large by only a single arithmetic operation. Note that the cancellation happens only in the addition and the subtraction, and the multiplication and the division do not cause the cancellation.

The most common cancellation by formula (2.7) happens in substituting a polynomial $h(v)$ for $x$ in $G(x,v)$. We may express the process of substitution roughly as

$$\begin{aligned} \sum_i g_i(v) h^i(v) \quad &\Rightarrow \quad \sum_e \Big( \sum_j \tilde{c}_{j,e} v^e \Big), \quad \tilde{c}_{j,e} \in \mathbf{C} \\ &= \quad \sum_e c_e v^e, \quad c_e = \sum_j \tilde{c}_{j,e}, \end{aligned}$$

where the upper and lower expressions show, respectively, the expressions before and after collecting the terms. If $\max\{|\tilde{c}_{j,e}| \,|\, j = 1, \ldots\} \gg |c_e|$ then we encounter the cancellation. The secondly common cancellation occurs when we shift the origin. Let $(s) \in \mathbf{C}^\ell$, and consider the transformation

$$P(x,u) \mapsto \tilde{P}(x,u) = P(x, u+s).$$

If $D = (|s_1|^2 + \cdots + |s_\ell|^2)^{1/2}$ is large then $\|\tilde{P}\|$ may be much different from $\|P\|$. For example, when $\|P\| \simeq 1$, $\deg(P) = 10$ and $D \simeq 10$ then we may have $\|\tilde{P}\| \approx 10^{10}$. Hence, if $\tilde{P}$ is given initially instead of $P$ then we encounter large cancellation errors in the computation of $\tilde{P}(x, u-s)$.

We call the polynomial $F(x,u)$ in (2.1) *regular* if it satisfies the following condition.

$$\max\{ \, \|f_{n-1}(u)\|, \; \cdots, \; \|f_0(u)\| \, \} = 1 \text{ or } 0. \tag{2.10}$$

Any polynomial $F(x,u)$ can be regularized easily by the scale transformation

$$F(x,u) \mapsto F(ax, bu), \quad a, b_1, \ldots, b_\ell \in \mathbf{C},$$

without changing the property of the root $\chi(u)$ essentially.

The analysis of cancellation becomes pretty complicated if we allow any kind of input polynomial. Therefore, in order to simplify the analysis, we restrict the input polynomials by imposing the following assumptions on $F(x,u)$.

**Assumption A** *The given polynomial $F(x,u)$ is such that its norm $\|F\|$ is made almost minimum by a suitable movement of the origin.*

**Assumption B** *The given polynomial $F(x,u)$ has already been regularized.*

## 2.3 On expansion at a singular point

Let us consider the case that $F(x,s)$ has multiple roots, then formula (2.7) breaks down if $\alpha$ is a multiple root of $F(x,s)$ hence $F'(\alpha,s)=0$.

**Definition 1 (singular point)** *Let* $R(u) = \mathrm{resultant}_x(F(x,u), F'(x,u))$ *and* $(s) \in \mathbf{C}^\ell$. *If* $R(s)=0$ *then the point* $(s)$ *is called* a singular point *of* $F(x,u)$.

**Remark 1** Note that $[(s)$ *is a singular point*$] \Longleftrightarrow [F(x,s)$ *has a multiple root*$]$. If $\alpha$ is a multiple root of $F(x,s)$ then $(\alpha,s)$ is a *singular point in the sense of algebraic geometry*; see [Wal78]. Even if $(s)$ is a singular point, $F(x,s)$ may have simple roots. Below, we assume that $\alpha$ is a multiple root of $F(x,s)$ when $(s)$ is a singular point. □

**Example 2** Singular points of $F(x,u)$ in Example 1.

$$\mathrm{resultant}_x(F,F') = -46656u^4(u-1)^3(64u^3-165u^2+192u-64)^2.$$

This resultant has the following five zero-points:

$$u=0,\quad 1,\quad 0.516926\cdots,\quad 1.030599\cdots \pm 0.934011\cdots i$$

Therefore, the expansion point in Example 1 is close to a singular point of $F(x,u)$. □

We can separate $F(x,u)$ to a singular factor $\hat{F}(x,u)$ and a nonsingular factor $\tilde{F}(x,u)$ in $\mathbf{C}\{u\}[x]$. Let $F(x,s) = \hat{F}(x,s)\tilde{F}(x,s)$ where $\tilde{F}(x,s)$ is square-free while $\hat{F}(x,s)$ has only multiple roots. Since $\gcd(\hat{F},\tilde{F}) = 1$, we can apply the generalized Hensel construction (see [GCL93]) to $F(x,u)$, with initial factors $\hat{F}(x,s)$ and $\tilde{F}(x,s)$, obtaining

$$F(x,u) = \hat{F}(x,u)\tilde{F}(x,u),\quad \hat{F}(x,u),\tilde{F}(x,u) \in \mathbf{C}\{u\}[x]. \tag{2.11}$$

$\hat{F}(x,u)$ is wholly singular at the point $(s)$ while $\tilde{F}(x,u)$ is not singular at $(s)$. Since $\hat{F}(x,s) = (x-\alpha_1)^{m_1}\cdots(x-\alpha_r)^{m_r}$, $\alpha_i \neq \alpha_j$ $(\forall i \neq j)$, we can factorize $\hat{F}(x,u)$ as

$$\left\{ \begin{array}{c} \hat{F}(x,u) = \hat{F}_1(x,u)\cdots\hat{F}_r(x,u), \\ \hat{F}_i(x,s) = (x-\alpha_i)^{m_i},\quad \hat{F}_i(x,u) \in \mathbf{C}\{u\}[x] \quad (i=1,\ldots,r). \end{array} \right. \tag{2.12}$$

For the bivariate polynomial $\hat{F}(x,u_1)$, we can expand the root $\hat{\chi}(u_1)$ at any singular point, obtaining a Puiseux series, i.e., a fractional-power series in $u_1$. For polynomials of more variables, Sasaki and Kako [SK99] found that, if we introduce the total-degree variable $t$ for the sub-variables, we can expand the roots at any singular point, obtaining (infinite) series which are of fractional-powers in $t$. Following [SK99], we describe the expansion method briefly. *In order to distinguish the expansion at a singular point from that at a nonsingular point, we attach* ˆ *to the expressions at the singular point.*

In the rest of this subsection, for simplicity, we assume that $r=1$ in (2.12) and put $\hat{F}_1(x,u) = \hat{F}(x,u)$. Furthermore, we assume without loss of generality that $(s)=(0)$ and $\alpha_1 = 0$. Hence, we have $\hat{F}(x,0) = x^m$. First, we introduce the total-degree variable $t$ by the mapping

$$u_i \mapsto tu_i \quad (i=1,\ldots,\ell). \tag{2.13}$$

(We may introduce the "weighted total-degree variable" by the mapping $u_i \mapsto t^{\omega_i}u_i$ $(i=1,\ldots,\ell)$, where $\omega_1,\ldots,\omega_\ell$ are nonnegative rational numbers.) Next, we introduce the concept of *Newton polynomial* as follows.

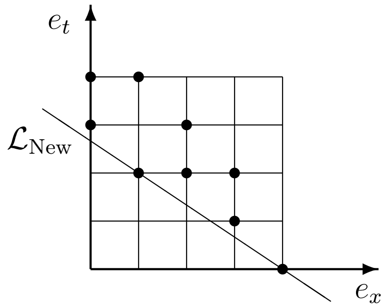


Fig. 1. Illustration of the Newton line

**Definition 2 (Newton line and Newton polynomial for $\hat{F}(x,u)$)**

1. *For each monomial $cx^it^ju_1^{j_1}\cdots u_\ell^{j_\ell}$ of $\hat{F}(x,u)$, with $c\in\mathbf{C}$ and $j=j_1+\cdots+j_\ell$, plot a dot at the point $(i,j)$ in the $(e_x,e_t)$-plane ;*
2. *Let $\mathcal{L}_{\rm New}$ be a straight line in $(e_x,e_t)$-plane, such that it passes the point $(m,0)$ and another dot plotted and that any dot plotted is not below $\mathcal{L}_{\rm New}$ ;*
3. *Construct $\hat{F}_{\rm New}(x,u)$ by summing all the monomials which are plotted on $\mathcal{L}_{\rm New}$ .*

*We call $\mathcal{L}_{\rm New}$ and $\hat{F}_{\rm New}(x,u)$ the* Newton line *and the* Newton polynomial*, respectively.*

Note that $\hat{F}_{\rm New}(x,u)$ is homogeneous w.r.t. $x$ and $t^\lambda$, where $\lambda$ is the slope of $\mathcal{L}_{\rm New}$.

**Example 3** The Newton polynomial for

$$\begin{aligned}\hat{P}(x,u_1,u_2) &= x^4+(u_1+2u_2+u_1u_2)x^3-(u_1^2+3u_1u_2^2)x^2\\ &\quad +(2u_1^4+u_1u_2-2u_2^2)x-(2u_1^3-3u_1^2u_2^2+5u_2^4).\end{aligned}$$

$\mathcal{L}_{\rm New}$ is a line passing the points $(4,0)$ and $(1,2)$, see Fig. 1. Collecting the terms plotted on $\mathcal{L}_{\rm New}$, we obtain $\hat{F}_{\rm New}(x,tu_1,tu_2)=x^4+t^2(u_1u_2-2u_2^2)x$. □

Following [SK99], we introduce the ideals $\hat{I}_k$ $(k=1,2,\ldots)$ as follows. Let $\mathcal{L}_{\rm New}$ be $e_x/m+e_t/\tau=1$ ($(0,\tau)$ is the intersection of $\mathcal{L}_{\rm New}$ and the $e_t$-axis), and let $\hat{m}$ and $\hat{\tau}$ be positive integers satisfying $\tau/m=\hat{\tau}/\hat{m}=\lambda$ and $\gcd(\hat{m},\hat{\tau})=1$, then

$$\hat{I}_k=\langle\, x^mt^{(k+0)/\hat{m}},\ x^{m-1}t^{(k+\hat{\tau})/\hat{m}},\ \cdots,\ x^0t^{(k+m\hat{\tau})/\hat{m}}\,\rangle. \tag{2.14}$$

According to Lemma 1 in [SK99], all the integer lattice points above $\mathcal{L}_{\rm New}$ are ridden by the generators of $\hat{I}_k$ $(k=1,2,\ldots)$.

Now, we determine the $k$-th order approximate "power series" root $\hat{\chi}^{(k)}(tu)$ satisfying

$$\hat{F}(\hat{\chi}^{(k)}(tu),tu)\equiv 0 \pmod{\hat{I}_{k+1}},\quad \mathrm{ord}_t(\hat{\chi}^{(k)})=\lambda+k/\hat{m}. \tag{2.15}$$

Let $\theta_1(u),\ldots,\theta_m(u)$ be the roots of $\hat{F}_{\rm New}(x,u)$:

$$\hat{F}_{\rm New}(x,u)=(x-\theta_1(u))\cdots(x-\theta_m(u)). \tag{2.16}$$

For simplicity, we assume that $\theta(u) \stackrel{\text{def}}{=} \theta_1(u)$ is a simple root of $\hat{F}_{\text{New}}(x, u)$ (we will remark on the case of multiple root at the end of this subsection). We note that $\hat{F}(x, tu) \equiv \hat{F}_{\text{New}}(x, tu) \pmod{\hat{I}_1}$ and $\hat{\chi}^{(0)}(tu) \stackrel{\text{def}}{=} t^\lambda \theta(u)$ is the zeroth order root of $\hat{F}(x, tu)$. Suppose we have computed $\hat{\chi}^{(k')}(tu) \stackrel{\text{def}}{=} \hat{\chi}_1^{(k')}(tu)$ $(k' = 0, \cdots, k-1)$, satisfying

$$\hat{F}(\hat{\chi}^{(k')}(tu), tu) \;\equiv\; 0 \pmod{\hat{I}_{k'+1}}, \quad k' = 0, \cdots, k-1.$$

Let the $k$-th order approximate roots be $\hat{\chi}_i^{(k)} = \hat{\chi}_i^{(k-1)} + \mathit{\Delta}\hat{\chi}_i^{(k)}$ $(i = 1, \ldots, m)$ :

$$\hat{F}(x, tu) \equiv (x - \hat{\chi}_1^{(k-1)} - \mathit{\Delta}\hat{\chi}_1^{(k)}) \cdots (x - \hat{\chi}_m^{(k-1)} - \mathit{\Delta}\hat{\chi}_m^{(k)}) \pmod{\hat{I}_{k+1}}.$$

Substituting $\hat{\chi}^{(k-1)}$ for $x$ in this equation and noting the order of $\hat{\chi}^{(k)}$, we obtain

$$\hat{F}(\hat{\chi}^{(k-1)}(tu), tu) \equiv -\mathit{\Delta}\hat{\chi}^{(k)}(tu) \cdot \hat{F}'_{\text{New}}(t^\lambda \theta, tu) \pmod{\hat{I}_{k+1}}. \tag{2.17}$$

We can rewrite the above formula to a form which is convenient to the computation. Let the square-free decomposition of $\hat{F}_{\text{New}}$ be

$$\begin{cases} \hat{F}_{\text{New}}(x, tu) = \hat{F}_{\text{N1}}(x, tu) \hat{F}_{\text{N2}}^2(x, tu) \cdots \hat{F}_{\text{Ns}}^s(x, tu), \\ \gcd(\hat{F}_{\text{Ni}}, \hat{F}_{\text{Nj}}) = 1 \quad \text{for any} \quad i \neq j. \end{cases} \tag{2.18}$$

Since $t^\lambda \theta$ is a simple root of $\hat{F}_{\text{New}}(x, tu)$, it must be a root of $\hat{F}_{\text{N1}}$, i.e. $\hat{F}_{\text{N1}}(t^\lambda \theta, tu) = 0$, and we have $\gcd(\hat{F}_{\text{N1}}(x, tu), \hat{F}'_{\text{New}}(x, tu)) = 1$. Hence, the extended Euclidean algorithm allows us to calculate $V(x, tu)$ and $W(x, tu)$ satisfying

$$\begin{cases} V(x, tu) \hat{F}_{\text{N1}}(x, tu) + W(x, tu) \hat{F}'_{\text{New}}(x, tu) = 1, \\ V(x, tu), W(x, tu) \in \mathbf{C}(tu)[x], \\ \deg(V) < m - 1, \quad \deg(W) < \deg(\hat{F}_{\text{N1}}). \end{cases} \tag{2.19}$$

Substituting $t^\lambda \theta$ for $x$ in the above first equation, we obtain

$$W(t^\lambda \theta, tu) \hat{F}'_{\text{New}}(t^\lambda \theta, tu) = 1. \tag{2.20}$$

Multiplying $W(t^\lambda \theta, tu)$ to (2.17) and using (2.20), we obtain

$$\hat{\chi}^{(k)}(tu) \;\equiv\; \hat{\chi}^{(k-1)}(tu) - W(t^\lambda \theta, tu) \hat{F}(\hat{\chi}^{(k-1)}(tu), tu) \pmod{t^{\lambda + (k+1)/\hat{m}}}. \tag{2.21}$$

This is our required formula.

**Theorem 1** *Let $\hat{F}(x, u)$ have a singular point at $(u) = (0)$, and $\theta$ be a simple root of $\hat{F}_{\text{New}}(x, u)$. The approximate power series root $\hat{\chi}^{(k)}(tu)$ $(k \geq 1)$, with $\hat{\chi}^{(0)}(tu) = t^\lambda \theta$, is of the form*

$$\hat{\chi}^{(k)}(tu) = t^\lambda \theta + \hat{y}_1(tu) + \cdots + \hat{y}_k(tu), \tag{2.22}$$

$$\text{ord}_t(\hat{y}_j(tu)) = \lambda + j/\hat{m} \quad (j = 1, 2, \ldots, k), \tag{2.23}$$

$$\hat{y}_j(u) \in \mathbf{C}\{(u)\}[\theta] \quad (j = 1, 2, \ldots, k). \tag{2.24}$$

*Proof.* We have shown (2.22) and (2.23) above.

We note that $V(x,u), W(x,u) \in \mathbf{C}(u)[x]$, because the extended Euclidean algorithm consists of rational operations. Furthermore, $V(x,tu)$ and $W(x,tu)$ are homogeneous w.r.t. $x$ and $t^\lambda$ because so are $\hat{F}_{\mathrm{N1}}(x,tu)$ and $\hat{F}'_{\mathrm{New}}(x,tu)$. Hence, each coefficient w.r.t. $x$, of $W(x,tu)$ is a homogeneous rational function in $u_1,\ldots,u_\ell$ and we see $W(\theta,u) \in \mathbf{C}\{(u)\}[\theta,t]$. Then, the iteration formula (2.21) reads us to (2.24). $\square$

**Corollary 1** *In the bivariate case (i.e., $\ell = 1$), if $\lambda$ is an integer (i.e., $\hat{F}_{\mathrm{New}}(x,u_1)$ is homogeneous w.r.t. $x$ and $u_1^{\hat{\tau}}$) then $\hat{\chi}^{(k)}(u_1)$ is a polynomial in $u_1$ for every $k$.*

*Proof.* Obvious becuase $\theta = \zeta u_1^{\hat{\tau}}$, $\zeta \in \mathbf{C}$ and $\mathbf{C}\{(u_1)\} = \mathbf{C}[u_1]$ in this case. $\square$

**Example 4** Series expansion at a singular point.

$$\hat{F}(x,u) \;=\; x^2 - 2(u_1+u_1^2)x + u_1^2 - u_2^2 - u_3^2 + 2u_1^3 + u_2^3 + u_3^3 + u_1^4. \tag{2.25}$$

$\hat{F}(x,u)$ has a singular point at the origin and the roots $\hat{\chi}_i(u)$ $(i=1,2)$ are

$$\begin{cases} \hat{\chi}_1(u) \;=\; u_1 + u_1^2 + \sqrt{u_2^2+u_3^2-u_2^3-u_3^3}, \\ \hat{\chi}_2(u) \;=\; u_1 + u_1^2 - \sqrt{u_2^2+u_3^2-u_2^3-u_3^3}. \end{cases}$$

We have $\hat{F}_{\mathrm{New}}(x,tu) = x^2 - 2tu_1x + t^2(u_1^2-u_2^2-u_3^2)$ which is square-free. Hence $\hat{F}_{\mathrm{N1}} = \hat{F}_{\mathrm{New}}$, and we obtain

$$V(x,tu) = \frac{-1}{t^2(u_2^2+u_3^2)}, \qquad W(x,tu) = \frac{x - tu_1}{2t^2(u_2^2+u_3^2)}.$$

Formula (2.21), as well as relation $\theta^2 - 2u_1\theta + (u_1^2-u_2^2-u_3^2) = 0$, gives us

$$\hat{\chi}^{(2)}(tu) \;=\; tu_1 + t^2u_1^2 + (t\theta - tu_1)\cdot\Big[1 - t\frac{u_2^3+u_3^3}{2(u_2^2+u_3^2)} + t^2\frac{(u_2^3+u_3^3)^2}{8(u_2^2+u_3^2)^2}\Big].$$

Using the explicit representation $\theta = u_1 + \sqrt{u_2^2+u_3^2}$, we obtain

$$\hat{\chi}^{(2)}(tu) \;=\; tu_1 + t^2u_1^2 + \sqrt{u_2^2+u_3^2}\cdot\Big[t - t^2\frac{u_2^3+u_3^3}{2(u_2^2+u_3^2)} + t^3\frac{(u_2^3+u_3^3)^2}{8(u_2^2+u_3^2)^2}\Big].$$

$\square$

We remark on the case that $\theta$ is a multiple root of $\hat{F}_{\mathrm{New}}$:

$$\hat{F}_{\mathrm{New}}(x,u) = (x-\theta)^\mu \hat{H}_{\mathrm{New}}(x,u), \quad \mu \geq 2. \tag{2.26}$$

Performing the extended Hensel construction of $\hat{F}_{\mathrm{New}}(x,tu)$ with initial factors $(x - t^\lambda\theta)^\mu$ and $\hat{H}_{\mathrm{New}}(x,tu)$, we obtain $\hat{G}^{(k)}(x,tu)$ and $\hat{H}^{(k)}(x,tu)$ $(k=1,2,\ldots)$ such that

$$\begin{cases} \hat{F}(x,tu) \equiv \hat{G}^{(k)}(x,tu)\hat{H}^{(k)}(x,tu) \pmod{\hat{I}_{k+1}}, \\ \hat{G}^{(0)}(x,u) = (x-\theta)^\mu, \quad \hat{H}^{(0)}(x,u) = \hat{H}_{\mathrm{New}}(x,u), \\ \hat{G}^{(k)}(x,u), \hat{H}^{(k)}(x,u) \in \mathbf{C}\{(u)\}[x,\theta]. \end{cases} \tag{2.27}$$

Let $\hat{G}^{(k)}(x,tu)$ be represented as

$$\hat{G}^{(k)}(x,tu) = x^{\mu} + \hat{g}_{\mu-1}(tu)x^{\mu-1} + \cdots + \hat{g}_0(tu)x^0. \tag{2.28}$$

Performing the following variable transformation

$$\hat{G}^{(k)}(x,tu) \longmapsto \check{G}^{(k)}(x,tu) = \hat{G}^{(k)}(x - \hat{g}_{\mu-1}(tu)/\mu, tu), \tag{2.29}$$

then the resulting $\check{G}^{(k)}(x,tu)$ satisfies

$$\check{G}^{(k)}(x,tu) \equiv x^{\mu} \pmod{\hat{I}_{k+1}}. \tag{2.30}$$

Therefore, the situation reduces to the original one: we can determine the Newton polynomial for $\check{G}^{(k)}(x,tu)$ and finally factorize it into factors which are linear in $x$.

## 2.4 Integral and non-integral root

Let us investigate the behavior of the approximate power series root near a singular point, which the reader will find very important for clarifying the cancellation error. In this subsection, we assume that the origin is a singular point of $F(x,u)$ and that the expansion point $(s)$ is inside the convergence radius of the series $\hat{\chi}^{(\infty)}(u)$. Therefore, $\hat{\chi}^{(k)}(s+v)$ can be expanded into a Taylor series in $v_1,\ldots,v_\ell$, and we have

$$\chi^{(\infty)}(v) \;=\; [\text{Taylor exansion of } \hat{\chi}^{(\infty)}(u) \text{ at } (u)=(s)]. \tag{2.31}$$

**Definition 3 (integral and non-integral root)** *Let $F(x,u)$ have a singular point at the origin, and $\hat{\chi}^{(k)}(u)$ be the $k$-th order approximate power series root of $F(x,u)$, expanded at the the origin. If $\hat{\chi}^{(k)}(u)$ can be expressed as*

$$\hat{\chi}^{(k)}(u) = \chi_0^{(k)}(u) + \hat{\chi}_1^{(k)}(\ldots,u_{l-1},u_{l+1},\ldots), \quad \chi_0^{(k)}(u) \in \mathbf{C}[u], \tag{2.32}$$

*then we say that the power series root $\hat{\chi}^{(k)}(u)$ is* integral *w.r.t. $u_l$ at the expansion point, otherwise we say that the root is* non-integral *w.r.t. $u_l$.*

For example, the roots $\hat{\chi}_i^{(2)}(u)$ $(i=1,2)$ in Example 4 are integral w.r.t. $u_1$ and non-integral w.r.t. $u_2$ and $u_3$.

**Remark 2** Corollary 1 implies that integral roots appear frequently in the bivariate case (the case of $\ell=1$). We can prove that a necessary and sufficient condition that $\hat{\chi}^{(\infty)}(u_1)$ becomes integral is that the slope $\lambda$ of the Newton line is an integer. □

The integral and non-integral roots show strikingly distinct behaviors near the singular point, as we will explain below. Let us consider a homogeneous rational function $R(u)$ or a homogeneous algebraic function $A(u)$ which may appear in the approximate power series root: $R(u)$ is in $\mathbf{C}\{(u)\}$ hence it can be expressed as $R(u)=N(u)/D(u)$ with $N(u)$ and $D(u)$ homogeneous polynomials of total-degrees $\kappa+\eta$ and $\kappa$, respectively; $A(u)$ is a root of $P_n(u)X^n + P_{n-1}(u)x^{n-1} + \cdots + P_0(u)$, where $P_i(u)$ $(i=n,\ldots,0)$ are homogeneous polynomials in $u_1,\ldots,u_\ell$, with $\mathrm{tdeg}(P_n)=\eta$, $\mathrm{tdeg}(P_{n-1})=\eta+\kappa$, $\cdots$, $\mathrm{tdeg}(P_0)=\eta+n\kappa$. Note that $R(tu)=t^{\kappa}R(u)$ and $A(tu)=t^{\kappa}A(u)$.

**Lemma 1** *Let $\hat{T}_\kappa(u)$ be either a rational function $R(u)$ or an algebraic function $A(u)$ defined above, hence $\mathrm{ord}_t(\hat{T}_\kappa(tu)) = \kappa$, and let $(s_1, \ldots, s_\ell) \in \mathbf{C}^\ell$ be not a singular point of $F(x, u)$. Let $\hat{T}_\kappa(s + v)$ be expanded into Taylor series in $v_1/s_1, \ldots, v_\ell/s_\ell$. Then, the coefficient $c(s)$ of each term $(v_1/s_1)^{e_1} \cdots (v_\ell/s_\ell)^{e_\ell}$ of the Taylor series is of order $\kappa$ w.r.t. $s_1, \ldots, s_\ell$.*

*Proof.* Being applied the transformation $s_i \mapsto ts_i$ and $v_i \mapsto tv_i$ $(i=1, \ldots, \ell)$, $\hat{T}_\kappa(s + v)$ is transformed as $\hat{T}_\kappa(s + v) \mapsto \hat{T}_\kappa(ts + tv) = t^\kappa \hat{T}_\kappa(s + v)$. Since $s_i + v_i = s_i \cdot (1 + v_i/s_i)$ $(i=1, \ldots, \ell)$, this transformation is equivalent to $s_i \mapsto ts_i$ and $v_i/s_i \mapsto v_i/s_i$. Applying the transformation to the Taylor expansion of $\hat{T}_\kappa(s + v)$, we see that the coefficient $c(s)$ is transformed as $c(s) \mapsto c(ts) = t^\kappa c(s)$. □

**Proposition 1** *Let $F(x, u)$ have a singular point at the origin, and let the expansion point $(s)$ be close to the orgin. Let $\hat{\chi}^{(k')}(u)$ and $\chi^{(k)}(v)$ be approximate power series roots, expanded at the origin and at $(u) = (s)$, respectively, where $\hat{\chi}^{(k')}(u)$ contains a term $\hat{T}_\kappa(u)$ such that $\mathrm{ord}_t(\hat{T}_\kappa(tu)) = \kappa$. If $\hat{T}_\kappa(u)$ is integral w.r.t. $x_l$ then $\kappa$ is an integer and $\hat{T}_\kappa(u)$ gives terms $T_0(s), T_1(v), \ldots, T_\kappa(v)$ in $\chi^{(\infty)}(v)$, such that $\|T_i(v)\| = O(\|s\|^{\kappa-i})$ $(i = 0, 1, \ldots, \kappa)$, hence $\hat{T}_\kappa(u)$ gives only a finite term $\hat{T}_\kappa(v)$ in the limit $\|s\| \to 0$. If $\hat{T}_\kappa(u)$ is non-integral w.r.t. $x_l$ then $\hat{T}_\kappa(u)$ gives infinitely many terms $T_0(s), T_1(v), T_2(v), \ldots$ in $\chi^{(\infty)}(v)$, such that $\|T_i(v)\| = O(\|s\|^{\kappa-i})$ $(i = 0, 1, 2, \ldots)$, hence the coefficients of $T_{\kappa+1}(v), T_{\kappa+2}(v), \ldots$ diverge as $\|s\| \to 0$.*

*Proof.* First, consider the the case that $\hat{T}_\kappa(u)$ is integral, hence it is a polynomial. Expanding $\hat{T}_\kappa(s + v)$ w.r.t. $v_1, \ldots, v_\ell$, we obtain the proposition directly.

Next, consider the case that $\hat{T}_\kappa(u)$ is non-integral, hence it is a rational function or an algebraic function investigated in Lemma 1. By expanding $\hat{T}_\kappa(s + v)$ w.r.t. $v_1, \ldots, v_\ell$, Lemma 1 leads us to the proposition directly. □

# 3 Analysis of terms of power series roots

In this section, we first rewrite the formula (2.7) to a form which allows us to express the $k$-th order term $y_k(v)$ by sub-polynomials of $F(x, s + v)$. Then, we investigate the leading terms of $y_k(v)$ theoretically.

For convenience, we rename the polynomial $F(x, s + v)$ as

$$G(x, v_1, \ldots, v_\ell) \stackrel{\text{def}}{=} F(x, s_1 + v_1, \ldots, s_\ell + v_\ell). \tag{3.1}$$

We decompose $G(x, v)$ into homogeneous polynomials w.r.t. $v_1, \ldots, v_\ell$, as

$$\begin{cases} G(x, v) = G_0(x) + G_1(x, v) + \cdots + G_\tau(x, v), \\ \quad G_j(x, v) = [G(x, v)]_j \quad (j = 0, 1, \ldots, \tau). \end{cases} \tag{3.2}$$

where $\tau$ is the total-degree w.r.t. $v_1, \ldots, v_\ell$, of $G$ :

$$\tau \;=\; \mathrm{tdeg}_v(G) = \mathrm{tdeg}_u(F). \tag{3.3}$$

We can calculate $G_j(x,v)$ by expanding $F(x,s+v)$ into Taylor series as

$$G_j(x,v) \;=\; \frac{1}{j!}\Big(v_1\frac{\partial}{\partial u_1}+\cdots+v_\ell\frac{\partial}{\partial u_\ell}\Big)^j F(x,u)\Big|_{(u)=(s)} \quad (j=0,\ldots,\tau). \tag{3.4}$$

As we will see below, $y_k(v)$ is composed of many terms of different magnitudes, so we define the dominant terms as follows.

**Definition 4 (dominant term)** *Let the approximate power series root be $\chi^{(k)}(v) = \alpha + y_1(v)+\cdots+y_k(v)$, as in (2.8), and let $y_k(v)$ be expressed as $y_k(v) = T_1(v)+T_2(v)+\cdots+T_\kappa(v)$. If $\|T_j\| = O(\|T_1\|)$* or *$o(\|T_1\|)$ for $j=2,\ldots,\kappa$ then $T_1$ is called the dominant term of $y_k(v)$. Next, suppose $y_k(v)$ is characterized by a real number $\rho$ such that*

$$\limsup_{k\to\infty}\ \|y_k(v)\| = \text{constant}\times(1/\rho)^{ak+b}, \tag{3.5}$$

*where $a$ and $b$ are constants. We call $y_k(v)$ a* dominant term *of $\chi^{(\infty)}(v)$ if its $\rho$-dependence is given by $\lim_{k\to\infty}\|y_k\| \propto (1/\rho)^{ak+b}$.*

## 3.1 Rewriting of Newton's formula

Now, we rewrite the formula (2.7). Substituting (2.8), with $k$ replaced by $k-1$, into the formula (2.7) and noting that $\mathrm{ord}_v(\chi^{(k)}-\chi^{(k-1)}) = k$, we obtain the following formula for $y_k(v)$ $(k\ge 1)$.

$$y_k(v) \;=\; -[G(\alpha+y_1+\cdots+y_{k-1},v)]_k/\beta,$$

where $\beta$ is a number defined by

$$\beta \;=\; G'(\alpha,0) = F'(\alpha,s). \tag{3.6}$$

Expanding $G(\alpha+y,v)$ into Taylor series in $y$, we obtain

$$\begin{aligned} y_k(v) \;=\; & -\ [G(\alpha,v)]_k/\beta \\ & -\ [G^{(1)}(\alpha,v)(y_1+\cdots+y_{k-1})^1]_k/1!\beta \\ & -\quad \cdot\ \cdot\ \cdot \\ & -\ [G^{(k)}(\alpha,v)(y_1+\cdots+y_{k-1})^k]_k/k!\beta, \end{aligned} \tag{3.7}$$

where $G^{(i)}$ is the $i$-th derivative of $G$ w.r.t. $x:\ G^{(i)} = \partial^i G/\partial x^i$.

In order to grasp the performance of formula (3.7), we calculate $y_1(v)$ and $y_2(v)$, for example. First, we calculate numbers $\alpha$ and $\beta$ by solving $G_0(x)=0$: $G_0(\alpha)=0$ and $\beta = G_0^{(1)}(\alpha)$. We can calculate $y_1(v)$ easily as

$$y_1(v) \;=\; -[G(\alpha,v)]_1/\beta = -G_1(\alpha,v)/\beta. \tag{3.8}$$

Then, $y_2(v)$ is calculated as follows.

$$\begin{aligned} y_2(v) \;&=\; -[G(\alpha,v)]_2/\beta - [G^{(1)}(\alpha,v)y_1(v)]_2/\beta - [G^{(2)}(\alpha,v)y_1(v)^2]_2/2\beta \\ &=\; -G_2(\alpha,v)/\beta - G_1^{(1)}(\alpha,v)y_1(v)/\beta - G_0^{(2)}(\alpha)y_1(v)^2/2\beta. \end{aligned}$$

Using (3.8), we can rewrite the above r.h.s. expression as

$$y_2(v) \;=\; -G_0^{(2)}(\alpha)G_1(\alpha,v)^2/2\beta^3 + G_1^{(1)}(\alpha,v)G_1(\alpha,v)/\beta^2 - G_2(\alpha,v)/\beta. \tag{3.9}$$

### 3.2 Basic lemma

**Lemma 2** *The $y_k(v)$ $(k \geq 1)$ is a sum of products of the form*

$$\gamma \, \{G_{l_0}^{(0)}(\alpha, v)\}^{e_0}\{G_{l_1}^{(r_1)}(\alpha, v)\}^{e_1} \cdots \{G_{l_\kappa}^{(r_\kappa)}(\alpha, v)\}^{e_\kappa}, \qquad \gamma \in \mathbf{C}, \tag{3.10}$$

*where $G_{l_j}^{(0)}(\alpha, v) = G_{l_j}(\alpha, v)$ and $G_{l_j}^{(r_j)}(\alpha, v) \neq G_0^{(1)}(\alpha)$ for any $j$, $r_i \neq r_j$ if $l_i = l_j$ for any $i \neq j$, and $0 \leq l_0, l_1, \ldots, l_\kappa \leq \min\{k, \tau\}$.* (Condition "$r_i \neq r_j$ if $l_i = l_j$" means that the same factors are collected.) *The numerical factor $\gamma$ in* (3.10) *is such that*

$$\gamma = \hat{\gamma}/\beta^e \quad \textit{with} \quad e = e_0 + e_1 + \cdots + e_\kappa \geq 1, \tag{3.11}$$

*where $\hat{\gamma}$ is independent of $\beta$. The $l_j$, $r_j$ and $e_j$ $(j = 0, \ldots, \kappa)$ satisfy*

$$\begin{aligned} l_0 e_0 + l_1 e_1 + \cdots + l_\kappa e_\kappa &= k, \qquad (3.12)\\ r_1 e_1 + \cdots + r_\kappa e_\kappa &= e - 1. \qquad (3.13) \end{aligned}$$

*Proof.* Eq. (3.8) shows that the first claim of 3emma 2 is valid. The claim is then a direct consequence of mathematical induction on index $k$ of formula (3.7). Since $\mathrm{tdeg}_v(G_0^{(1)}(\alpha)(y_1 + \cdots + y_{k-1})) = k - 1$ in (3.7), $G_0^{(1)}(\alpha)$ does not contribute to $y_k$, which means $G_{l_j}^{(r_j)}(\alpha, v) \neq G_0^{(1)}(\alpha) = \beta$ for any $j$ in (3.10).

Formula (3.7) shows that each $G_{l_j}^{(r_j)}(\alpha, v)$ factor in (3.10) is accompanied with one $(1/\beta)$ factor. Hence, we obtain (3.11).

Since $\mathrm{tdeg}_v(G_{l_j}^{(r_j)}(\alpha, v)) = l_j$ and $\mathrm{tdeg}_v(y_k) = k$, we obtain (3.12).

Eq. (3.13) is valid for $y_1$, as (3.8) shows. Suppose that it is valid for $y_1, \ldots, y_{k-1}$, $k \geq 2$, and consider the $r$-th term $[G^{(r)}(\alpha, v)(y_1 + \cdots + y_{k-1})^r]_k / r!\beta$ in the r.h.s. of (3.7). This term gives a sum of products of the form

$$\text{constant} \times [G_l^{(r)}(\alpha, v)/\beta] \; y_{k_1}^{e_1} \cdots y_{k_r}^{e_r}, \tag{3.14}$$

where $e_1 + \cdots + e_r = r$ and $l + k_1 e_1 + \cdots + k_r e_r = k$. We assume that $r \geq 1$, because, for $r = 0$, (3.14) becomes $-G_k^{(0)}(\alpha, v)/\beta$ and we have $l = k$, $e_0 = e = 1$, and $e_j = 0$ $(j \geq 1)$, hence (3.13) becomes $0 = 0$. We note that the l.h.s. of (3.13) shows the number of differentiations w.r.t. $x$, and $e$ is the exponent of $(1/\beta)$, as (3.11) shows. Hence, by the induction assumption, we see

$$\begin{aligned} & [\#\text{ of differentiations}] - [\text{exponent of } (1/\beta)] \;\text{ in }\; y_{k_1}^{e_1} \cdots y_{k_r}^{e_r} \\ = & \textstyle\sum_{j=1}^{r} \Big( [\#\text{ of differentiations}] - [\text{exponent of } (1/\beta)] \;\text{ in }\; y_{k_j} \Big) \times e_j \\ = & \textstyle\sum_{j=1}^{r} (-1) \times e_j = -(e_1 + \cdots + e_r) = -r. \end{aligned}$$

On the other hand, $G_l^{(r)}(\alpha, v)$ factor in (3.14) gives $r$ differentiations and another $(1/\beta)$ factor. Hence, Eq. (3.13) is valid for $y_k$, too. □

### 3.3 $\beta$-dependence of $y_k$ : frequent case

By the frequent case, we mean that we have $G_0^{(2)}(\alpha) \neq 0$ and $G_1(\alpha, v) \neq 0$.

**Lemma 3** *If $G_0^{(2)}(\alpha) \neq 0$ and $G_1(\alpha, v) \neq 0$, the $\beta$-dependence of $y_k$ is such that*

$$y_k \;=\; \hat{y}_{k,2k-1}/\beta^{2k-1} + \hat{y}_{k,2k-2}/\beta^{2k-2} + \cdots + \hat{y}_{k,1}/\beta, \tag{3.15}$$

*where $\hat{y}_{k,j}$ $(j = 2k-1, 2k-2, \ldots, 1)$ are independent of $\beta$. Furthermore,*

$$\hat{y}_{k,2k-1} \;=\; -\hat{\gamma}_k\, G_0^{(2)}(\alpha)^{k-1} G_1(\alpha, v)^k \quad \textit{with} \quad \hat{\gamma}_k > 0, \tag{3.16}$$

*where $\hat{\gamma}_1 = 1$, $\hat{\gamma}_2 = 1/2$, and $\hat{\gamma}_k$ $(k = 3, 4, \ldots)$ are calculated as*

$$\hat{\gamma}_k \;=\; (\hat{\gamma}_1\hat{\gamma}_{k-1} + \hat{\gamma}_2\hat{\gamma}_{k-2} + \cdots + \hat{\gamma}_{k-1}\hat{\gamma}_1)/2. \tag{3.17}$$

*Proof.* Eq. (3.15) is valid for $y_1$. Suppose that it is valid for $y_1, \ldots, y_{k-1}$, $k \geq 2$, and consider the powers of $(1/\beta)$ in the r.h.s. of (3.7). The term $[G(\alpha, v)]_k/\beta$ gives only $(1/\beta)^1$, and $[G^{(1)}(\alpha, v)(y_1 + \cdots + y_{k-1})^1]_k/1!\beta$ gives $(1/\beta)^e$, with $1 \leq e \leq 2k-2$. Note that, since the product in (3.10) contains no $G_0^{(1)}(\alpha)$ factor, as Lemma 2 says, and $G_0^{(1)}(\alpha) = \beta$ by definition, each $\beta$ factor in the denominator is not cancelled by any factor in the numerator. Next, for $r \geq 2$, consider the term $[G_l^{(r)}(\alpha, v)(y_1 + \cdots + y_{k-1})^r]_k/r!\beta$. This term gives a sum of products of the form in (3.14). Induction assumption tells us that the product gives the largest $(1/\beta)$-power as

$$\begin{aligned} & G_l^{(r)} y_1^{e_1} \cdots y_r^{e_r}/\beta \quad \text{with } e_1 + \cdots + e_r = r \\ \Rightarrow\; & (1/\beta)^{1+(2k_1-1)e_1+\cdots+(2k_r-1)e_r} = (1/\beta)^{2(k-l)-r+1}, \end{aligned}$$

because $l + k_1e_1 + \cdots + k_re_r = k$. Hence, the largest $(1/\beta)$-power in $y_k$ appears only from the term with the smallest values of $r$ and $l$ (i.e., $r = 2$ and $l = 0$), and the largest power is $(1/\beta)^{2k-1}$. Therefore, (3.15) is valid for $y_k$, too.

Eq. (3.16) is valid for $y_1$ and $y_2$. Suppose that it is valid for $y_1, \ldots, y_{k-1}$, $k \geq 3$. Then, the above consideration shows that the term $\hat{y}_{k,2k-1}/\beta^{2k-1}$ in (3.15) is given by the term $[G_0^{(2)}(\alpha)(y_1 + \cdots + y_{k-1})^2]_k/2!\beta$, and induction assumption tells us that

$$\begin{aligned} & [G_0^{(2)}(\alpha)(y_1 + \cdots + y_{k-1})^2]_k/2!\beta \\ = \;& -G_0^{(2)}(\alpha)(y_1y_{k-1} + y_2y_{k-2} + \cdots + y_{k-1}y_1)/2\beta \\ \Rightarrow \;& -G_0^{(2)}(\alpha)\Big\{\, \hat{\gamma}_1\hat{\gamma}_{k-1}\, G_1(\alpha, v) \cdot G_0^{(2)}(\alpha)^{k-2}\, G_1(\alpha, v)^{k-1} \\ & \qquad + \hat{\gamma}_2\hat{\gamma}_{k-2}\, G_0^{(2)}(\alpha)\, G_1(\alpha, v)^2 \cdot G_0^{(2)}(\alpha)^{k-3}\, G_1(\alpha, v)^{k-2} + \;\cdot\;\cdot\;\cdot\; \Big\}\Big/2\beta^{2k-1} \\ = \;& -\{\hat{\gamma}_1\hat{\gamma}_{k-1} + \hat{\gamma}_2\hat{\gamma}_{k-2} + \cdots + \hat{\gamma}_{k-1}\hat{\gamma}_1\} \times G_0^{(2)}(\alpha)^{k-1}\, G_1(\alpha, v)^k/2\beta^{2k-1}. \end{aligned}$$

This shows that (3.16) with (3.17) is valid for $y_k$, too. $\square$

## 3.4 When $G_1(x, v) = 0$

In some case, $G_1(\alpha, v), \ldots, G_{\nu-1}(\alpha, v)$ may become zero. In such a case, Lemma 3 must be modified. For example, if $G_1^{(2)}(\alpha) \neq 0$, $G_1(x, v) = 0$ and $G_2(\alpha, v) \neq 0$, we have $y_1(v) = 0$ and

$$\begin{aligned}
y_2(v) &= -G_2(\alpha, v)/\beta, \\
y_3(v) &= -G_3(\alpha, v)/\beta, \\
y_4(v) &= -G_0^{(2)}(\alpha) G_2(\alpha, v)^2/2\beta^3 + G_2^{(1)}(\alpha, v) G_2(\alpha, v)/\beta^2 - G_4(\alpha, v)/\beta, \\
y_5(v) &= -G_0^{(2)}(\alpha) G_2(\alpha, v) G_3(\alpha, v)/\beta^3 + G_2^{(1)}(\alpha, v) G_3(\alpha, v)/\beta^2 + \ \cdots .
\end{aligned}$$

In this subsection, we investigate a restricted case that $G_1(x, v) = \cdots = G_{\nu-1}(x, v) = 0$, $2 \leq \nu \leq \tau$. Note that this case often occurs practically. We, furthermore, assume that $G_\nu(\alpha, v) \neq 0$, because if $G_1(\alpha, v) = \cdots = G_\tau(\alpha, v) = 0$ then $G(x, v) = (x - \alpha)\tilde{G}(x, v)$ and $G(x, v)$ contains the trivial root $\chi(v) = \alpha$.

**Lemma 4** *Let $2 \leq \nu \leq \tau$. In the case that $G_0^{(2)}(\alpha) \neq 0$,*

$$G_1(x, v) = \cdots = G_{\nu-1}(x, v) = 0 \quad \textit{and} \quad G_\nu(\alpha, v) \neq 0, \tag{3.18}$$

*the numerical factor $\gamma$ in (3.10) is such that*

$$\gamma = \hat{\gamma}/\beta^e \quad \textit{with} \quad 1 \ \leq \ e = e_0 + e_1 + \cdots + e_\kappa \ \leq \ 2[k/\nu] - 1, \tag{3.19}$$

*where $\hat{\gamma}$ is independent of $\beta$ and $[\xi]$, with $\xi$ a real number, is Gauss' notation showing the largest integer not greater than $\xi$. The $\beta$-dependence of $y_k$ $(k \geq \nu)$ is such that*

$$y_k \ = \ \hat{y}_{k,2[k/\nu]-1}/\beta^{2[k/\nu]-1} + \hat{y}_{k,2[k/\nu]-2}/\beta^{2[k/\nu]-2} + \cdots + \hat{y}_{k,1}/\beta, \tag{3.20}$$

*where $\hat{y}_{k,j}$ $(j = 2[k/\nu] - 1, \ldots, 1)$ are independent of $\beta$. Furthermore,*

$$\hat{y}_{k,2[k/\nu]-1} \ = \ -\hat{\gamma}_{[k/\nu]} \, G_0^{(2)}(\alpha)^{[k/\nu]-1} \Big[ (G_\nu + \cdots + G_{2\nu-1})^{[k/\nu]} \Big]_k , \tag{3.21}$$

*where $\hat{\gamma}_l$ $(l = 1, 2, \ldots)$ are the same as those in Lemma 3.*

*Proof.* We put $l = [k/\nu]$. Applying formula (3.7), we find

$$y_1(v) = \cdots = y_{\nu-1}(v) = 0, \tag{3.22}$$

$$y_j(v) = -G_j(\alpha, v)/\beta \quad (j = \nu, \ldots, 2\nu - 1). \tag{3.23}$$

Hence, (3.20) and (3.21) are valid for $y_1, \ldots, y_\nu, \ldots, y_{2\nu-1}$. Suppose that they are valid for $y_1, \ldots, y_\nu, \ldots, y_{l\nu-1}$, $l \geq 2$, and consider $y_k$ where $l\nu \leq k < (l+1)\nu$. Then, in formula (3.7), the term $[G(\alpha, v)]_k/\beta$ gives only $(1/\beta)^1$, and $[G^{(1)}(\alpha, v)(y_\nu + \cdots + y_{k-1})^1]_k/1!\beta$ gives $(1/\beta)^e$, with $1 \leq e \leq 2l - 2$. In order to see this, we note that

$$[G^{(1)}(\alpha, v)(y_\nu + \cdots + y_{k-1})^1]_k = [G^{(1)}(\alpha, v)(y_\nu + \cdots + y_{k-\nu})^1]_k,$$

because $G_1(x,v) = \cdots = G_{\nu-1}(x,v) = 0$. Thus, we see $e \le 2[(k-\nu)/\nu] - 1 \le 2l-2$ by induction assumption. Investigating the product in (3.14) as in Lemma 3, we see that the largest $(1/\beta)$-power appears only from $[G_0^{(2)}(\alpha,v)(y_\nu + \cdots + y_{k-1})^2]_k/2!\beta$, as

$$G_0^{(2)}(\alpha)\Big\{(y_\nu y_{(l-1)\nu+k'} + \cdots + y_{\nu+k'}y_{(l-1)\nu}) \; + \\ (y_{2\nu}y_{(l-2)\nu+k'} + \cdots + y_{2\nu+k'}y_{(l-2)\nu}) \; + \; \cdots\Big\}\Big/2\beta,$$

where we have put $k = l\nu + k'$ $(0 \le k' < \nu)$. (Terms $y_{\nu+k'+k''}y_{(l-1)\nu-k''}$, with $1 \le k''$ and $k'+k'' < \nu$, give smaller $(1/\beta)$-powers.) Induction assumption tells us that this expression gives $(1/\beta)^{2[k/\nu]-1}$ factor, hence (3.20) is valid for $y_{l\nu}, \ldots, y_{l(\nu+1)-1}$, too. Substituting (3.21), with $k$ replaced by $i\nu + j$ $(1 \le i \le l,\ 0 \le j \le k')$, for $y_{i\nu+j}$ in the $i$-th $(\cdots)$-term of the above brace, we find

$$\begin{aligned} & (y_{i\nu}y_{(l-i)\nu+k'} + \cdots + y_{i\nu+k'}y_{(l-i)\nu}) \\ \propto \; & \hat\gamma_i\hat\gamma_{l-i} \textstyle\sum_{j=0}^{k'} \Big[(G_\nu + \cdots + G_{2\nu-1})^i\Big]_{i\nu+j} \cdot \Big[(G_\nu + \cdots + G_{2\nu-1})^{l-i}\Big]_{(l-i)\nu+k'-j} \\ = \; & \hat\gamma_i\hat\gamma_{l-i} \Big[(G_\nu + \cdots + G_{2\nu-1})^l\Big]_{l\nu+k'} . \end{aligned}$$

Hence, (3.21) with (3.17) is valid for $y_k$, too. □

**Remark 3** Eq. (3.20) shows that, if the largest $(1/\beta)$-power term dominates $y_k(v)$, the power series $\chi(v)$ is dominated by $\{y_\nu, y_{2\nu}, \ldots, y_{l\nu}, \ldots\}$. Although the expression in (3.21) is complicated, $y_{l\nu}(v)$ is quite simple as follows.

$$y_{l\nu} \;=\; -\hat\gamma_l\, G_0^{(2)}(\alpha)^{l-1} G_\nu(\alpha,v)^l/\beta^{2l-1} + \text{smaller } (1/\beta)\text{-power terms.} \tag{3.24}$$

## 3.5 Rare case : $G_0^{(2)}(\alpha) = 0$

If $G_0^{(2)}(\alpha) = 0$ then $y_k(v)$ shows a different $(1/\beta)$-dependence. For example, when $G_1(\alpha,v) \ne 0$, $G_0^{(2)}(\alpha) = 0$ and $G_0^{(3)}(\alpha) \ne 0$, we have

$$\begin{aligned} y_1(v) &= -G_1(\alpha,v)/\beta, \\ y_2(v) &= G_1^{(1)}(\alpha,v)G_1(\alpha,v)/\beta^2 - G_2(\alpha,v)/\beta, \\ y_3(v) &= G_0^{(3)}(\alpha)G_1(\alpha,v)^3/6\beta^4 - G_1^{(1)}(\alpha,v)^2G_1(\alpha,v)/\beta^3 + \;\cdots, \\ y_4(v) &= -\{2G_0^{(3)}(\alpha)G_1^{(1)}(\alpha,v)G_1(\alpha,v)^3/3 + G_0^{(4)}(\alpha)G_1(\alpha,v)^4/24\}/\beta^5 + \;\cdots. \end{aligned}$$

Note that, since we assumed that $G_0^{(1)}(\alpha) = \beta \ne 0$, the case $G_0^{(2)}(\alpha) = 0$ or $|G_0^{(2)}(\alpha)| \ll |G_0^{(3)}(\alpha)|$ occurs only accidentally. In this subsection, for completeness of our analysis, we investigate this case. Note that this is the most general case.

**Lemma 5** *Let $\mu \ge 2$ and $1 \le \nu \le \tau$. In the case that* (3.18) *holds and*

$$G_0^{(2)}(\alpha) = \cdots = G_0^{(\mu)}(\alpha) = 0 \quad \textit{and} \quad G_0^{(\mu+1)}(\alpha) \ne 0, \tag{3.25}$$

*the $\beta$-dependence of $y_k$ $(k \ge \nu)$ is such that*

$$y_k \;=\; \hat y_{k,[k/\nu]+[(k-\nu)/\mu\nu]}/\beta^{[k/\nu]+[(k-\nu)/\mu\nu]} + \text{smaller } (1/\beta)\text{-power terms,} \tag{3.26}$$

*where* $\hat{y}_{k,j}$ $(j = [k/\nu] + [(k-\nu)/\mu\nu],\ \ldots,\ 1)$ *are independent of* $\beta$. *Furthermore, for* $k = (l\mu+1)\nu$ $(l = 0, 1, \ldots)$, *the largest* $(1/\beta)$*-power term is given by*

$$\hat{y}_{(l\mu+1)\nu,\, l(\mu+1)+1}/\beta^{l(\mu+1)+1} \ = \ -\hat{\gamma}_l \,\{G_0^{(\mu+1)}(\alpha)\}^l \, G_\nu(\alpha,v)^{l\mu+1}/\beta^{l(\mu+1)+1}, \tag{3.27}$$

*where* $\hat{\gamma}_0 = 1$, $\hat{\gamma}_1 = (-1)^{\mu+1}/(\mu+1)!$, *and* $\hat{\gamma}_l$ $(l \geq 2)$ *are calculated as*

$$\hat{\gamma}_l \ = \ (-1)^{\mu+1}\Big[(\hat{\gamma}_0 + \hat{\gamma}_1 + \cdots + \hat{\gamma}_{l-1})^{\mu+1}\Big]_{\text{index sum}=l-1}\Big/(\mu+1)!, \tag{3.28}$$

*where* $[\cdots]_{\text{index sum}=l-1}$ *denotes the sum of all the terms* $\hat{\gamma}_{l_0}\cdots\hat{\gamma}_{l_\mu}$, *with* $l_0+\cdots+l_\mu = l-1$, *in the expansion of* $(\hat{\gamma}_0 + \cdots + \hat{\gamma}_{l-1})^{\mu+1}$.

*Proof.* Let $\tilde{y}_k(v)$ be the sum of all the terms of $y_k(v)$, containing no $G_0^{(i)}(\alpha)$ factor, $(i = 0, 1, 2, \ldots)$. We first show that the largest $(1/\beta)$-power of $\tilde{y}_k$ is $[k/\nu]$.

Condition (3.18) gives (3.22) and (3.23), regardless of the value of $G_0^{(2)}(\alpha)$. Hence, the above claim is valid for $\tilde{y}_1, \ldots, \tilde{y}_\nu, \ldots, \tilde{y}_{2\nu-1}$. Suppose that the claim is valid for $\tilde{y}_1, \ldots, \tilde{y}_\nu, \ldots, \tilde{y}_{l\nu-1}$, $l \geq 2$, and consider $\tilde{y}_k$ where $l\nu \leq k < (l+1)\nu$. Then, in formula (3.7), the term $[G(\alpha,v)]_k/\beta$ gives only $(1/\beta)^1$, and $[G^{(1)}(\alpha,v)(\tilde{y}_\nu + \cdots + \tilde{y}_{k-1})^1]_k/1!\beta$ gives $(1/\beta)^e$, with $1 \leq e \leq [(k-\nu)/\nu] + 1 = [k/\nu]$. Similarly, for any $i \geq 2$, the term $[G^{(i)}(\alpha,v)(\tilde{y}_\nu + \cdots + \tilde{y}_{k-1})^i]_k/i!\beta$ also gives the largest $(1/\beta)$-power terms of $\tilde{y}_k$ and we can see by induction assumption that the largest power is $(1/\beta)^{[k/\nu]}$.

Now, let us consider $[G_0^{(i)}(\alpha)(y_\nu + \cdots)^i]_k$, where $i \geq \nu+1$. This term gives nonzero contribution to $y_k$ only when $i\nu \geq k$. Hence, for $k \leq (\mu+1)\nu - 1$, we have $y_k(v) = \tilde{y}_k(v)$. However, for $k = (\mu+1)\nu$, $G_0^{(\mu+1)}(\alpha)$ contributes to $y_{(\mu+1)\nu}$ as follows.

$$\begin{aligned} y_{(\mu+1)\nu} \ &\Leftarrow\ -\Big[G^{(\mu+1)}(\alpha,v)(y_\nu + \cdots)^{\mu+1}\Big]_{(\mu+1)\nu}\Big/(\mu+1)!\beta \\ &=\ -\, G_0^{(\mu+1)}(\alpha)\; y_\nu^{\mu+1}/(\mu+1)!\beta - \ \cdots \\ &=\ -\,(-1)^{\mu+1}\, G_0^{(\mu+1)}(\alpha)\; G_\nu(\alpha,v)^{\mu+1}/(\mu+1)!\;\beta^{\mu+2} - \ \cdots . \end{aligned}$$

Note that the largest $(1/\beta)$-power in $\tilde{y}_{(\mu+1)\nu}$ is $\mu+1$. Therefore, the largest $(1/\beta)$-power term in $y_{(\mu+1)\nu}$ is a term containing $G_0^{(\mu+1)}(\alpha)$.

Next, consider the case of $(\mu+1)\nu \leq k \leq (2\mu+1)\nu - 1$. Since $[G_0^{(i)}(\alpha)(y_\nu + \cdots + y_{k-1})^i]_k = [G_0^{(i)}(\alpha)(y_\nu + \cdots + y_{k-(i-1)\nu})^i]_k$ and $i \geq \mu+1$, only $y_k, \ldots, y_{(\mu+1)\nu-1}$ contribute to $y_k$. Hence, the largest $(1/\beta)$-power in $y_k$ is $[k/\nu]+1$. However, for $k = (2\mu+1)\nu$, we obtain $(1/\beta)^{[k/\nu]+2}$ term because the product $y_\nu^\mu\, y_{(\mu+1)\nu}$ contributes to $y_{(2\mu+1)\nu}$ :

$$\begin{aligned} y_{(2\mu+1)\nu} \ &\Leftarrow\ -\Big[G^{(\mu+1)}(\alpha,v)(y_\nu + \cdots + y_{(2\mu+1)\nu-1})^{\mu+1}\Big]_{(2\mu+1)\nu}\Big/(\mu+1)!\beta \\ &=\ -\, G_0^{(\mu+1)}(\alpha)\cdot(\mu+1)\; y_\nu^\mu y_{(\mu+1)\nu}\Big/(\mu+1)!\beta - \ \cdots \\ &=\ -\,\{G_0^{(\mu+1)}(\alpha)\}^2\; G_\nu(\alpha,v)^{2\mu+1}/(\mu+1)!\;\mu!\;\beta^{2\mu+3} - \ \cdots . \end{aligned}$$

By mathematical induction, Eq. (3.26) can be proved similarly.

Finally, let us consider (3.27) with (3.28). For general value of $k$, expression of the largest $(1/\beta)$-power term in $y_k$ is complicated because many different kinds of terms contribute to it. For $k = (l\mu+1)\nu \ \ (l = 1, 2, \ldots)$, however, we can calculate it as follows.

$$\begin{aligned}
& \Big[G_0^{(\mu+1)}(\alpha)\ (y_\nu + y_{(\mu+1)\nu} + \cdots + y_{((l-1)\mu+1)\nu})^{\mu+1}\Big]_{(l\mu+1)\nu} \\
= \ & G_0^{(\mu+1)}(\alpha)\Big\{ {}_{\mu+1}C_{\mu,1}\ y_\nu^\mu y_{((l-1)\mu+1)\nu} \ + \ {}_{\mu+1}C_{\mu-1,1,1}\ y_\nu^{\mu-1} y_{(\mu+1)\nu} y_{((l-2)\mu+1)\nu} \\
& \qquad + \ \cdots \ + \ {}_{\mu+1}C_{\mu-2,2,1}\ y_\nu^{\mu-2} y_{(\mu+1)\nu}^2 y_{((l-3)\mu+1)\nu} \ + \ \cdots \Big\},
\end{aligned}$$

where ${}_mC_{m_1,\ldots,m_i}$, with $m = m_1 + \cdots + m_i$, is a multinomial coefficient : ${}_mC_{m_1,\ldots,m_i} = m!/(m_1! \cdots m_i!)$. Each product in the above brace gives the largest $(1/\beta)$-power term of the following same form.

$$\text{constant} \times \{G_0^{(\mu+1)}(\alpha)\}^{l-1}\ G_\nu(\alpha, v)^{l\mu+1}/\beta^{l(\mu+1)}.$$

Therefore, considering the numerical factor $-\hat{\gamma}_l$, we obtain (3.27) with (3.28). $\square$

**Remark 4** Contrary that $\hat{\gamma}_k$ in (3.16) and $\hat{\gamma}_{[k/\nu]}$ in (3.21) are positive, the $\hat{\gamma}_l$ in (3.27) changes its sign when $\mu$ is an even integer. However, it is easy to verify that, when $\mu$ is even, we have $\hat{\gamma}_l = (-1)^l|\hat{\gamma}_l|$ and, in the calculation of the r.h.s. expression in (3.28), only the products $\hat{\gamma}_{l_0} \cdots \hat{\gamma}_{l_\mu}$ of the same sign are added. That is, there happens no cancellation during the calculation. For example, when $\mu = 2$, we have $\hat{\gamma}_0 = 1$, $\hat{\gamma}_1 = -1/6$, and

$$\begin{aligned}
\hat{\gamma}_2 &= -[(\hat{\gamma}_0 + \hat{\gamma}_1)^3]_{\text{index sum}=1}/6 = -3\hat{\gamma}_0^2\hat{\gamma}_1/6 = 1/12, \\
\hat{\gamma}_3 &= -[(\hat{\gamma}_0 + \hat{\gamma}_1 + \hat{\gamma}_2)^3]_{\text{index sum}=2}/6 = -(3\hat{\gamma}_0^2\hat{\gamma}_2 + 3\hat{\gamma}_0\hat{\gamma}_1^2)/6 \\
&= -(1/24 + 1/72) = -1/18, \\
\hat{\gamma}_4 &= -[(\hat{\gamma}_0 + \cdots + \hat{\gamma}_3)^3]_{\text{index sum}=3}/6 = -(3\hat{\gamma}_0^2\hat{\gamma}_3 + 6\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2 + \hat{\gamma}_1^3)/6 \\
&= -(-1/36 - 1/72 - 1/1296) = 55/1296.
\end{aligned}$$

# 4 Order estimation of cancellation errors

In this section, by $G_j^{(i)}$*-product* we mean a product in (3.10) with general indices $i$ and $j$. Furthermore, we use the following notations.

$$\left\{ \begin{array}{c} F(x, u) = x^n + f_{n-1}(u)x^{n-1} + \cdots + f_0(u)x^0, \\ \tau_i = \mathrm{tdeg}_u(f_i(u)) \quad (i = n-1, \ldots, 0). \end{array} \right. \tag{4.1}$$

$$\left\{ \begin{array}{rcl} G_0(x) &=& x^n + f_{n-1}(s)x^{n-1} + \cdots + f_0(s)x^0, \\ G_j(x, v) &=& g_{j,n-1}(v)x^{n-1} + \cdots + g_{j,0}(v)x^0 \quad (j = 1, \ldots, \tau). \end{array} \right. \tag{4.2}$$

In discussing the cancellation, it is convenient to separate the process of computation of $y_k(v)$ into two steps, as follows.

**Step 1:** The expression in formula (3.7) is expanded, and $y_k(v)$ is expressed as a sum of $G_j^{(i)}$-products in (3.10), where the $G_j^{(i)}$-products are not expanded;

**Step 2:** The $G_j^{(i)}$-products constructed in Step 1 are expanded and collected.

In Step 1, serious cancellation may be caused only by the substitution of $\alpha$ for $x$ in $G_j^{(i)}(x, v)$, which we call *cancellation by $\alpha$-substitution.* We call the cancellation which may appear in Step 2 *cancellation among $G_j^{(i)}$-products.*

## 4.1 Cancellation errors by $\alpha$-substitution

At first, we note that the effect of the cancellation by $\alpha$-substitution can be compensated by computing $\alpha$, $\beta$, $G_0^{(i)}(\alpha)$ and $G_j^{(i)}(\alpha,v)$ with an increased precision.

Let us first consider the cancellation in $G^{(i)}(x)$ by $\alpha$-substitution. Suppose we have the cancellation by $\alpha$-substitution for $i=1$:

$$\begin{aligned} & 0 \neq |\beta| = |G_0^{(1)}(\alpha)| \ll B, \quad \text{where} \\ & B \;=\; \max\{\; n|\alpha^{n-1}|,\; (n{-}1)|\alpha^{n-2}f_{n-1}(s)|,\; \cdots,\; 1|\alpha^0 f_1(s)| \;\}. \end{aligned} \tag{4.3}$$

Here, $B$ is the maximum magnitude term w.r.t $\alpha$, of $G_0^{(1)}(\alpha)$. Since $\beta$ appears in every $G_j^{(i)}$-product in $y_k$ $(k=1,2,\ldots)$, the effect of this cancellation is quite large. Note that, even if we have (4.3), there is some term $G_0^{(i)}$ $(1<i\le n)$ for which no cancellation occurs. For example, for $i=n$, we have $G_0^{(n)}(x)=G_0^{(n)}(\alpha)=n!$, because $G(x,v)$ is monic. Suppose we encounter large cancellations by $\alpha$-substitution for $G_0^{(1)}(x),\ldots,G_0^{(m-1)}(x)$, but no large cancellation for $G_0^{(m)}(x)$. Then, we can express $G_0(x)$ as

$$\left\{ \begin{aligned} & G_0(x) = (x-\alpha)^m \tilde{G}_0(x) + \mathit{\Delta} G_0(x), \\ & \qquad \|\mathit{\Delta} G_0(x)\| \ll \|\tilde{G}_0(x)\|. \end{aligned} \right. \tag{4.4}$$

Relations in (4.4) show that $G_0(x)$ has $m$ close roots around $x=\alpha$, or we may say that $G_0(x)$ has $m$ *approximate multiple roots* at $x=\alpha$, which suggests us that there is a singular point near the expansion point.

Let us consider what amount of cancellation error occurs in $G_j^{(i)}(\alpha,v)$ by $\alpha$-substitution when the expansion point is close to a singular point. Suppose that $(\hat{\alpha},\hat{s})$ is a singular point of multiplicity $m$, in the sense of algebraic geometry, hence we have (see [Wal78]) hence we have

$$\frac{\partial^{i_0}}{\partial x^{i_0}} \frac{\partial^{i_1}}{\partial u_1^{i_1}} \cdots \frac{\partial^{i_\ell}}{\partial u_\ell^{i_\ell}} F(x,u)\Big|_{x=\hat{\alpha},\,(u)=(\hat{s})} \;=\; 0 \quad \text{for} \;\; i_0+i_1+\cdots+i_\ell \le m-1. \tag{4.5}$$

Suppose further that the expansion point $(s)$ is close to $(\hat{s})$ so that we have

$$\|s-\hat{s}\| = (|s_1{-}\hat{s}_1|^2+\cdots+|s_\ell{-}\hat{s}_\ell|^2)^{1/2} \;\ll\; \|F(x,s)\|. \tag{4.6}$$

Formula (3.4) tells us that we can calculate $G_j^{(i)}(\alpha,v)$ as

$$G_j^{(i)}(\alpha,v) \;=\; \frac{1}{j!}\Big[v_1\frac{\partial}{\partial u_1}+\cdots+v_\ell\frac{\partial}{\partial u_\ell}\Big]^j \frac{\partial^i}{\partial x^i} F(x{-}\hat{\alpha}{+}\alpha,u{-}\hat{s}{+}s)\Big|_{x=\hat{\alpha},\,(u)=(\hat{s})}. \tag{4.7}$$

On the other hand, we have

$$\begin{aligned} F(x{-}\hat{\alpha}{+}\alpha,u{-}\hat{s}{+}s) \;&=\; F(x,u) \;+ \\ \sum_{j'=1}^{\max\{n,\tau\}} &\frac{1}{j'!}\Big[(\alpha{-}\hat{\alpha})\frac{\partial}{\partial x}+(s_1{-}\hat{s}_1)\frac{\partial}{\partial u_1}+\cdots+(s_\ell{-}\hat{s}_\ell)\frac{\partial}{\partial u_\ell}\Big]^{j'} F(x,u). \end{aligned} \tag{4.8}$$

Substituting this expression for $F(x-\hat{\alpha}+\alpha, u-\hat{s}+s)$ in (4.7), we can express $G_j^{(i)}(\alpha, v)$ in terms of $\frac{\partial^{i_0}}{\partial x^{i_0}}\frac{\partial^{i_1}}{\partial u_1^{i_1}}\cdots\frac{\partial^{i_\ell}}{\partial u_\ell^{i_\ell}}F(x,u)\Big|_{x=\hat{\alpha},\,(u)=(\hat{s})}$'s. Therefore, putting $\hat{B}_{i,j}$ as

$$\hat{B}_{i,j} \;=\; \max\Big\{\, \frac{(n-1)!}{(n-i-1)!}|\hat{\alpha}^{n-i-1}g_{j,n-1}(\hat{s})|,\ \frac{(n-2)!}{(n-i-2)!}|\hat{\alpha}^{n-i-2}g_{j,n-2}(\hat{s})|,\ \cdots\,\Big\}, \tag{4.9}$$

we obtain from (4.5) and (4.8) the following order estimation.

$$\frac{\|G_j^{(i)}(\alpha,v)\|}{B_{i,j}} \;=\; \begin{cases} O(1), & i+j\ge m,\\ O(\max\{|\alpha-\hat{\alpha}|/|\hat{\alpha}|,\ \|s-\hat{s}\|/\|\hat{s}\|\}^{m-(i+j)}), & i+j<m. \end{cases} \tag{4.10}$$

Therefore, the $\alpha$-substitution for $G_j^{(i)}(x,v)$, with $i+j<m$, causes the cancellation errors of magnitude $O(\max\{|\alpha-\hat{\alpha}|/|\hat{\alpha}|,\ \|s-\hat{s}\|/\|\hat{s}\|\}^{m-(i+j)})$.

## 4.2 $\|G_j^{(i)}$-product$\|$ near a singular point

By "near a singular point", we mean that the distance between the expansion point and the closest singular point is much less than 1, because $F$ has already been regularized by Assumptions A and B. We assume that the origin is a singular point, as before. In this subsection, we consider the order dependence of $\|G_j^{(i)}$-product$\|$ on $\|s\|$, by treating $\|s\|$ as an order parameter.

Let $F(x,u)=\hat{F}(x,u)\tilde{F}(x,u)$ as in (2.11), where $\hat{F}(x,u),\tilde{F}(x,u)\in \mathbf{C}\{u\}[x]$, $\hat{F}(x,0)=x^m$, and $\tilde{F}(x,u)$ is not singular at $(u)=(0)$. Furthermore, let the slope of the Newton line for $\hat{F}(x,u)$ be $\lambda$, hence $\hat{F}(x,u)$ satisfies

$$\begin{cases} \hat{F}(x,u)=x^m+x^{m-1}\hat{f}_{m-1}(u)+\cdots+x^0\hat{f}_0(u),\\ [\hat{f}_{m-1}(u)]_0^{\lambda-1}=[\hat{f}_{m-2}(u)]_0^{2\lambda-1}=\cdots=[\hat{f}_0(u)]_0^{m\lambda-1}=0,\\ [\hat{f}_{m-1}(u)]_\lambda+[\hat{f}_{m-2}(u)]_{2\lambda}+\cdots+[\hat{f}_0(u)]_{m\lambda}\ne 0, \end{cases} \tag{4.11}$$

where $[\hat{f}_{m-j}(u)]_{j\lambda}=0$ if $j\lambda$ is not an integer. Then, the coefficients of $F(x,u)$ satisfy

$$\begin{cases} [f_n(u)]_0\ne 0, \quad [f_m(u)]_0\ne 0,\\ [f_{m-1}(u)]_0^{\lambda-1}=[f_{m-2}(u)]_0^{2\lambda-1}=\cdots=[f_0(u)]_0^{m\lambda-1}=0,\\ [f_{m-1}(u)]_\lambda+[f_{m-2}(u)]_{2\lambda}+\cdots+[f_0(u)]_{m\lambda}\ne 0. \end{cases} \tag{4.12}$$

Let the expansion point $(s_1,\dots,s_\ell)$ be such that

$$\delta \overset{\text{def}}{=} \|s\| = (|s_1|^2+\cdots+|s_\ell|^2)^{1/2} \ll 1. \tag{4.13}$$

In order to discuss the cancellation, we must estimate the magnitudes of $\|G_j^{(i)}$-product$\|$. Note that the magnitudes of coefficients of $G_0(x)$ and $G_j(x,v)$ are

$$\begin{cases} |f_i(s)| &=& O(\delta^{\max\{0,\,(m-i)\lambda\}}) \quad (i=0,\dots,n-1),\\ \|g_{j,i}(v)\| &=& O(\delta^{\max\{0,\,(m-i)\lambda-j\}}) \quad (j=1,2,\dots). \end{cases} \tag{4.14}$$

(Precisely speaking, we may have $|f_i(s)| = o(\|s\|^{\tau_i})$ if $\|f_i(u)\| \ll 1$ or the cancellation happens in the calculation of $f_i(s)$. Then, we redefine $\lambda$ so that (4.14) holds.)

We choose $\alpha$ to be a root of $\hat{F}(x,s)$, because $\tilde{F}(x,u)$ is not singular at the origin. Therefore, we estimate the magnitudes of $|\alpha|$ and $|\beta|$ as follows.

$$\begin{cases} \hat{G}_0(\alpha) \stackrel{\text{def}}{=} \hat{F}(\alpha,s) = 0 & \Longrightarrow \quad |\alpha| = O(\delta^{\lambda}), \\ \beta = \hat{F}^{(1)}(\alpha,s)\tilde{F}(\alpha,s) & \Longrightarrow \quad |\beta| = O(\delta^{(m-1)\lambda}), \end{cases} \tag{4.15}$$

because $\hat{F}(\alpha,s) = 0$ and $\tilde{F}(\alpha,s) = O(\delta^0)$. Using (4.14), we can estimate the magnitude of $\|G_j^{(i)}(\alpha,v)\|$ as follows.

$$\|G_j^{(i)}(\alpha,v)\| = O(\delta^{\max\{0,\,(m-i)\lambda-j\}}) \quad (i=0,1,\ldots;\ j=0,1,\ldots). \tag{4.16}$$

For small $i$ and $j$, (4.15) and (4.16) give $\|G_j^{(i)}(\alpha,v)/\beta\| = O(\delta^{(1-i)\lambda-j})$.

**Proposition 2** *The $G_j^{(i)}$-product in* (3.10) *is order estimated as follows.*

$$\|\ G_j^{(i)}\text{-product}\ \| = \begin{cases} O(\delta^{\lambda-k}) & \text{if the product is dominant,} \\ o(\delta^{\lambda-k}) & \text{if the product is not dominant.} \end{cases} \tag{4.17}$$

*Proof.* Suppose the $G_j^{(i)}$-product contains only such $G_j^{(i)}$ factors that satisfy $(m-i)\lambda - j \ge 0$. Then, (4.16) tells us that

$$\begin{aligned} & \|\ \hat{\gamma}\ \{G_{j_0}^{(0)}/\beta\}^{e_0}\{G_{j_1}^{(i_1)}/\beta\}^{e_1}\cdots\{G_{j_\kappa}^{(i_\kappa)}/\beta\}^{e_\kappa}\| \\ = \ & O(\hat{\gamma})\,O(\delta^{(1-0)\lambda-j_0})^{e_0}\,O(\delta^{(1-i_1)\lambda-j_1})^{e_1}\cdots O(\delta^{(1-i_\kappa)\lambda-j_\kappa})^{e_\kappa} \\ = \ & O(\hat{\gamma})\,O(\delta^{(\Sigma e_r)\lambda-(\Sigma i_r e_r)\lambda-(\Sigma j_r e_r)}) = O(\hat{\gamma})\,O(\delta^{\lambda-k}). \end{aligned}$$

Here, the summations are over all the $G_{j_r}^{(i_r)}$ factors in the product, hence $\Sigma_{r=1}^{\kappa} e_r = e$ by (3.11), $\Sigma_{r=1}^{\kappa} i_r e_r = e-1$ by (3.13), and $\Sigma_{r=1}^{\kappa} j_r e_r = k$ by (3.12).

Now, consider the largest $(1/\beta)$-power terms analyzed in **3.3** $\sim$ **3.5**. Each of the terms contains only such $G_j^{(i)}$ factors that satisfy $(m-i)\lambda - j \ge 0$. Furthermore, the coefficient of the term is not zero, and we see $\hat{\gamma} = O(\delta^0)$. Therefore, we see $\|G_j^{(i)}\text{-product}\| = O(\delta^{\lambda-k})$ for such a term. On the other hand, many $G_j^{(i)}$-products contain such $G_j^{(i)}$ factors that satisfy $(m-i)\lambda - j < 0$. For such $G_j^{(i)}$-products, we have $\|G_j^{(i)}\text{-product}\| = o(\delta^{\lambda-k})$. $\square$

## 4.3 Cancellation errors near a singular point

The order estimation of $G_j^{(i)}$-product in the previous subsection leads us to the following remarkable theorem on the cancellation errors near a singular point.

As before, by $\hat{\chi}^{(k)}(u)$ and $\chi^{(k)}(v)$ we denote approximate power series roots expanded at a singular point and at $(u) = (s)$, respectively.

**Theorem 2** *Let $F(x,u)$ have a singular point and the expansion point be close to the singular point. Let $\delta$ denote the distance between the singular point and the expansion* *point*: $0 < \delta \ll 1$. *Let* $\hat{T}_\kappa(u)$, *with* $\mathrm{ord}_t(\hat{T}_\kappa(tu)) = \kappa$, *be the lowest order term that is non-integral w.r.t. some variable, of* $\hat{\chi}^{(\infty)}(u)$, *and let* $T_k(v)$, *with* $\mathrm{ord}_t(T_k(tv)) = k$, *be a dominant term of* $\chi^{(k)}(v)$. *Then, for* $k < \kappa$, *there occur cancellation errors of magnitude* $O(\delta^{\lambda-k})$ *in* $T_k(v)$. *For* $k \geq \kappa$, *the largest cancellation errors that occur in* $T_k(v)$ *are of magnitude* $O(\delta^{\lambda-\kappa})$.

*Proof.* For simplicity, we assume that the origin is the singular point. Hence, the Taylor expansion of $\hat{\chi}^{(\infty)}(u)$ at $(u) = (s)$ gives $\chi^{(\infty)}(v)$ and we have $\delta = (|s_1|^2 + \cdots + |s_\ell|^2)^{1/2} \ll 1$.

Case of $k < \kappa$. By assumption, any term $\hat{T}_k(u)$ of order $k$, of $\hat{\chi}^{(\infty)}(u)$ is a polynomial and we have $\|\hat{T}_k(u)\| = O(\delta^0)$. Hence, Prop. 1 tells us that $\hat{T}_k(s+v) = \hat{T}_k(v) + T'_k(v)$, $\|T'_k\| = O(\delta)$. On the other hand, Prop. 2 tells us that $\|$dominant $G_j^{(i)}$-product$\| = O(\delta^{\lambda-k})$. Hence, there must occur cancellation among dominant $G_j^{(i)}$-products, and $T_k(v)$ suffers cancellation errors of magnitude $O(\delta^{\lambda-k})$.

Case of $k \geq \kappa$. Since $\hat{T}_\kappa(u)$ is non-integral w.r.t. some variable, Prop. 1 tells us that $\hat{T}_\kappa(s+v)$ gives infinitely many polynomial terms $T_0(s), T_1(v), T_2(v), \cdots$ such that $\mathrm{tdeg}_v(T_i(v)) = i$ and $\|T_i(v)\| = O(\delta^{\kappa-i})$ $(i = 0, 1, 2, \ldots)$. Prop. 1 also tells us that the lowest order non-integral term of $\hat{\chi}^{(\infty)}(u)$ gives the dominant terms of $\chi^{(\infty)}(v)$. Comparing this fact with Prop. 2, we see that any dominant term $T_k(v)$ in $\chi^{(\infty)}(v)$ suffers cancellation errors of magnitude $O(\delta^{\lambda-\kappa})$. □

**Corollary 2** *If* $\hat{\chi}^{(\infty)}(u)$ *is integral w.r.t.* $x_l$ *then any dominant term* $T_k(v)$ *that contains* $x_l$ *suffers cancellation errors of magnitude* $O(\delta^{\lambda-k})$. *If the lowest order term of* $\hat{\chi}^{(\infty)}(u)$ *is non-integral w.r.t.* $x_l$ *then any dominant term* $T_k(v)$ *that contains* $x_l$ *suffers only cancelltion errors of magnitude* $O(\delta^0)$.

In **2.2**, we have shown an example of occurrence of extremely large errors. It corresponds to the case of an integral root and $\delta \approx 10^{-4}$. Hence, Theorem 2 tells us that we encounter cancellation errors of magnitude $\sim O(10^{4k})$ in dominant terms of $y_k(v)$. In the rest of this subsection, we check the above theorem by simple examples.

**Example 5** Case of non-integral root of a bivariate polynomial.

$$\begin{cases} F(x,u) = x^4 + c_3 u x^3 + c_2 u x^2 + c_1 u^2 x + c_0 u^2, \\ \qquad c_i = O(\delta^0) \quad (i = 3, \ldots, 0). \end{cases}$$

Note that the origin is a singular point of $F(x,u)$ and the power series roots are non-integral at the origin: $\hat{\chi}^{(k)}(u) = \zeta u^{1/2} + \cdots$, $\zeta \in \mathbf{C}$. Putting $u = \delta + v$, we calculate $y_1(v)$ and $y_2(v)$. The $\alpha$ and $\beta$ are determined as

$$G_0(\alpha) = \alpha^4 + \alpha^3\delta c_3 + \alpha^2\delta c_2 + \alpha\delta^2 c_1 + \delta^2 c_0 = 0 \implies |\alpha| = O(\delta^{1/2}),$$
$$\beta = G_0^{(1)}(\alpha) = 4\alpha^3 + 3\alpha^2\delta c_3 + 2\alpha\delta c_2 + \delta^2 c_1 \implies |\beta| = O(\delta^{3/2}).$$

Similarly, $G_j^{(i)}(\alpha, v)$ $(1 \leq i+j \leq 2)$ are determined as

$$\begin{array}{lcll} G_0^{(2)}(\alpha) & = & 12\alpha^2 + 6\alpha\delta c_3 + 2\delta c_2 & \implies \|G_0^{(2)}\| = O(\delta^1), \\ G_1^{(0)}(\alpha, v) & = & (\alpha^3 c_3 + \alpha^2 c_2 + 2\alpha\delta c_1 + 2\delta c_0)v & \implies \|G_1^{(0)}\| = O(\delta^1), \\ G_1^{(1)}(\alpha, v) & = & (3\alpha^2 c_3 + 2\alpha c_2 + 2\delta c_1)v & \implies \|G_1^{(1)}\| = O(\delta^{1/2}), \\ G_2^{(0)}(\alpha, v) & = & (\alpha c_1 + c_0)v^2 & \implies \|G_2^{(0)}\| = O(\delta^0). \end{array}$$

Substituting the above expressions into (3.8), we obtain

$$y_1(v) \;=\; -\frac{\alpha^2 c_2 + 2\delta c_0 + \text{h.o.t.}}{4\alpha^3 + 2\alpha\delta c_2 + \text{h.o.t.}} v \implies \|y_1(v)\| = O(\delta^{-1/2}),$$

where h.o.t. denotes higher order terms. Next, we calculate $\beta^3\times$[r.h.s. of (3.9)].

$$\begin{aligned}
\beta^3 y_2(v) \;&=\; -G_0^{(2)}(\alpha)G_1^{(0)}(\alpha,v)^2/2 + \beta G_1^{(1)}(\alpha,v)G_1^{(0)}(\alpha,v) - \beta^2 G_2^{(0)}(\alpha,v) \\
&=\; G_0(\alpha)\Big\{\alpha^2(2c_2^2 - 16c_0) + \delta(c_2^3 - 8c_2c_0) + \text{h.o.t.}\Big\}v^2 \\
&\quad + \Big\{\alpha^2\delta^2(-c_2^4 + 6c_2^2c_0 - 8c_0^2) + \delta^3(-c_2^3c_0 + 4c_2c_0^2) + \text{h.o.t.}\Big\}v^2 \\
&=\; -\Big\{\delta^2[\alpha^2(c_2^2 - 2c_0) + \delta c_2c_0](c_2^2 - 4c_0) + \text{h.o.t.}\Big\}v^2,
\end{aligned}$$

because $G_0(\alpha) = 0$. Each term in the r.h.s. of (3.9) is of magnitude $O(\delta^{-3/2})$ in the present case, and we see that the dominant terms in $y_k$ (i.e., $O(\delta^3)$ terms in $\beta^3 y_2$) do not cancel each other, and we obtain

$$y_2(v) \;\simeq\; -\frac{\delta^2[\alpha^2(c_2^2 - 2c_0) + \delta c_2c_0](c_2^2 - 4c_0)}{(4\alpha^3 + 2\alpha\delta c_2)^3} v^2 \implies \|y_2(v)\| = O(\delta^{-3/2}).$$

We note that $\|y_2(v)\|/\|y_1(v)\| = O(\delta^{-1})$. □

**Example 6** Case of integral root of a bivariate polynomial.

$$\begin{cases} F(x,u) \;=\; x^2 + (c_{11}u + c_{12}u^2)x + (c_{02}u^2 + c_{03}u^3), \\ \qquad |c_{1,1+j}|, |c_{0,2+j}| = O(\delta^0) \quad (j = 0,1). \end{cases}$$

Note that the origin is a singular point of $F(x,u)$ and the power series roots are integral at the origin: $\hat{\chi}^{(k)}(u) = \zeta u + \cdots$, $\zeta \in \mathbf{C}$. Putting $u = \delta + v$, we calculate $y_1(v)$ and $y_2(v)$. The $\alpha$ and $\beta$ are determined as

$$\begin{aligned}
&G_0(\alpha) = \alpha^2 + \alpha\delta(c_{11} + \delta c_{12}) + \delta^2(c_{02} + \delta c_{03}) = 0 &&\implies |\alpha| = O(\delta), \\
&\beta = G_0^{(1)}(\alpha) = 2\alpha + \delta c_{11} + \delta^2 c_{12} &&\implies |\beta| = O(\delta).
\end{aligned}$$

Similarly, $G_j^{(i)}(\alpha,v)$ $(1 \le i+j \le 2)$ are calculated as

$$\begin{aligned}
G_0^{(2)}(\alpha) \;&=\; 2 &&\implies \|G_0^{(2)}\| = O(\delta^0), \\
G_1^{(0)}(\alpha,v) \;&=\; [\alpha(c_{11} + 2\delta c_{12}) \\
&\qquad + \delta(2c_{02} + 3\delta c_{03})]v &&\implies \|G_1^{(0)}\| = O(\delta^1), \\
G_1^{(1)}(\alpha,v) \;&=\; (c_{11} + 2\delta c_{12})v &&\implies \|G_1^{(1)}\| = O(\delta^0), \\
G_2^{(0)}(\alpha,v) \;&=\; (c_{02} + \alpha c_{12} + 3\delta c_{03})v^2 &&\implies \|G_2^{(0)}\| = O(\delta^0).
\end{aligned}$$

Substituting the above expressions into (3.8), we obtain

$$y_1(v) \;=\; -\frac{\alpha c_{11} + 2\delta c_{02} + \text{h.o.t.}}{2\alpha + \delta c_{11} + \delta^2 c_{12}} v \implies \|y_1(v)\| = O(\delta^0),$$

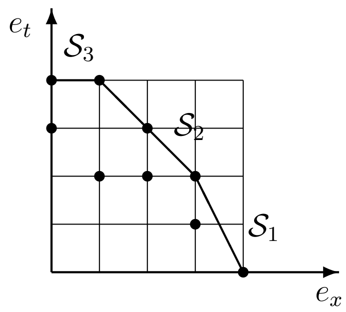


Fig. 2. Illustration of convex hull $\Omega$

where h.o.t. denotes higher order terms. Next, we calculate $\beta^3\times$[r.h.s. of (3.9)].

$$\begin{aligned}
\beta^3 y_2(v) &= -G_0^{(2)}(\alpha)G_1^{(0)}(\alpha,v)^2/2 + \beta G_1^{(1)}(\alpha,v)G_1^{(0)}(\alpha,v) - \beta^2 G_2^{(0)}(\alpha,v) \\
&= G_0(\alpha)\Big\{(c_{11}^2 - 4c_{02}) + 4(-\alpha c_{12} + \delta c_{11}c_{12} - 3\delta c_{03}) + 4\delta^2 c_{12}^2\Big\}v^2 \\
&\quad + \Big\{\alpha\delta^2(4c_{12}c_{02} - c_{11}^2 c_{12}) + \delta^3(4c_{02}c_{03} - c_{11}^2 c_{03}) + \text{h.o.t.}\Big\}v^2 \\
&= \Big\{\delta^2(\alpha c_{12} + \delta c_{03})(4c_{02} - c_{11}^2) + \text{h.o.t.}\Big\}v^2,
\end{aligned}$$

because $G_0(\alpha) = 0$. Although each term in the r.h.s. of (3.9) is of magnitude $O(\delta^{-1})$ in the present case, the dominant terms in $y_k$ (i.e., $O(\delta^2)$ terms in $\beta^3 y_2$) cancel one another completely, and we obtain

$$y_2(v) \simeq \frac{\delta^2(\alpha c_{12} + \delta c_{03})(4c_{02} - c_{11}^2)}{(2\alpha + \delta c_{11} + \delta^2 c_{12})^3}v^2 \implies \|y_2(v)\| = O(\delta^0).$$

We note that the mechanism of cancellation is considerably complicated. $\square$

## 4.4 Expansion at a distant point

Let the expansion point $(s_1, \ldots, s_\ell)$ be far from the origin, hence we have

$$D \stackrel{\text{def}}{=} \|s\| = (|s_1|^2 + \cdots + |s_\ell|^2)^{1/2} \gg 1. \tag{4.18}$$

Then, $f_i(s+v)$ $(i = 0, 1, \ldots, n)$ are approximated as

$$f_i(s+v) \approx [f_i(u)]_{\tau_i}\Big|_{(u)=(s)+(v)}, \quad i = 0, 1, \ldots, n. \tag{4.19}$$

Thus, the coefficients of $G_0(x)$ and $G_j(x,v)$ in (4.2) are order estimated as

$$|f_i(s)| = O(D^{\tau_i}), \quad \|g_{j,i}(v)\| = O(D^{\tau_i - j}) \ \ (j \le \tau_i). \tag{4.20}$$

Using (4.20), we order estimate $\alpha$ and $\beta$ in the following way.

Let us plot each monomial of $F(x, tu)$ in the $(e_x, e_t)$-plane, as in Definition 2, and let $\Omega$ be the convex hull of the dotts plotted. Let $\mathcal{S}_1, \ldots, \mathcal{S}_\sigma$ be the upper sides of $\Omega$, counted from the right to the left, as in Fig. 2. Let $-\bar{\lambda}_1, \ldots, -\bar{\lambda}_\sigma$ be the slopes of $\mathcal{S}_1, \ldots, \mathcal{S}_\sigma$, respectively (hence $\lambda_1 > 0$). Let $F_{\mathcal{S}_1}(x,u), \ldots, F_{\mathcal{S}_\sigma}(x,u)$ be the polynomials obtained, respectively, by summing all the terms plotted on $\mathcal{S}_1, \ldots, \mathcal{S}_\sigma$. Since $D \gg 1$, we have

$$\begin{aligned} F(x,s+v) \ \approx\ & F_{\mathcal{S}_1}(x,s+v) + F_{\mathcal{S}_2}(x,s+v) + \cdots + F_{\mathcal{S}_\sigma}(x,s+v) \\ & -\mathrm{lt}(F_{\mathcal{S}_2}(x,s+v)) - \cdots - \mathrm{lt}(F_{\mathcal{S}_\sigma}(x,s+v)), \end{aligned} \tag{4.21}$$

$$\text{where } \ \mathrm{lt}(F_{\mathcal{S}_i}) \text{ is the leading term of } F_{\mathcal{S}_i} \quad (i = 2, \ldots, \sigma).$$

Note that, for each $i = 1, \ldots, \sigma$, $F_{\mathcal{S}_i}(x,tu)$ is homogeneous w.r.t. $x$ and $t^{-\bar{\lambda}_i}$. Then, due to the convexity of $\Omega$, $\alpha$ satisfies one of the following approximate equalities.

$$\left\{ \begin{array}{lll} F_{\mathcal{S}_1}(\alpha,s) \approx 0 & \Longrightarrow & \alpha = \alpha_1 = O(D^{\bar{\lambda}_1}), \\ \quad \cdot\ \cdot\ \cdot\ \cdot & & \qquad \cdots \\ F_{\mathcal{S}_\sigma}(\alpha,s) \approx 0 & \Longrightarrow & \alpha = \alpha_\sigma = O(D^{\bar{\lambda}_\sigma}). \end{array} \right. \tag{4.22}$$

We note that $|\alpha_1| \gg \cdots \gg |\alpha_\sigma|$. In the rest of this subsection, we choose $\alpha = \alpha_\iota$ where $1 \le \iota \le \sigma$ and $|\alpha_\iota| \gg 1$, and put $\bar{\lambda} = \bar{\lambda}_\iota$. Hence, we have

$$\alpha = O(D^{\bar{\lambda}}), \quad \bar{\lambda} > 0. \tag{4.23}$$

$F_{\mathcal{S}_\iota}(x,s)$ may have "close" roots (by "close" roots we mean that their mutual distances are much smaller than $|\alpha|$). In this subsection, in order to make the argument clear, we assume that $\alpha$ *is not such a close root* (we will remove this restriction in **5.2**). Then, (4.21) and the convexity of $\Omega$ tell us that $\beta = F'(\alpha,s)$ is dominated by $F'_{\mathcal{S}_\iota}(\alpha,s)$, and we have

$$\beta = O(D^{(\bar{n}-1)\bar{\lambda}+\bar{\tau}}), \quad \bar{n} = \deg_x(F_{\mathcal{S}_\iota}(x,u)), \quad \bar{\tau} = \tau_{\bar{n}}. \tag{4.24}$$

(The factor $D^{\bar{\tau}}$ comes because $F_{\mathcal{S}_\iota}(x,s+v) \approx f_{\bar{n}}(s+v)x^{\bar{n}} + f_{\bar{n}-1}(s+v)x^{\bar{n}-1} + \cdots$ ).

Next, we order estimate $\|G_j^{(i)}(\alpha,v)\|$. We note that $F_{\mathcal{S}_\iota}(x,u)$ contributes to dominant terms of $G_j^{(i)}(x,v)$ so long as $(\bar{n}-i)\bar{\lambda} - j \ge 0$. On the other hand, the terms which are not plotted on $\mathcal{S}_\iota$ contribute to non-dominant $G_j^{(i)}(\alpha,v)$'s for which we may have $(\bar{n}-i)\bar{\lambda} - j < 0$. Therefore, we find the following order estimation of $\|G_j^{(i)}(\alpha,v)\|$.

$$\|G_j^{(i)}(\alpha,v)\| = O(D^{\max\{0,\,(\bar{n}-i)\bar{\lambda}-j\}+\bar{\tau}}). \tag{4.25}$$

For small $i$ and $j$, (4.24) and (4.25) give $\|G_j^{(i)}(\alpha,v)/\beta\| = O(D^{(1-i)\bar{\lambda}-j})$.

**Proposition 3** *Let the expansion point be far from the origin*: $D = (|s_1|^2 + \cdots + |s_\ell|^2)^{1/2} \gg 1$. *Let $\alpha$ be determined to satisfy $F_{\mathcal{S}_\iota}(\alpha,s) \approx 0$, $\alpha = O(D^{\bar{\lambda}})$ where $\bar{\lambda} = \lambda_\iota$. Furthermore, let $\alpha$ be separated from other roots of $F(x,s)$ by the distance $O(D^{\bar{\lambda}})$ or more. Then, the $G_j^{(i)}$-product in* (3.10) *is order estimated as follows.*

$$\|\ G_j^{(i)}\text{-product}\ \| = \left\{ \begin{array}{ll} O(D^{\bar{\lambda}-k}) & \text{if the product is dominant,} \\ o(D^{\bar{\lambda}-k}) & \text{if the product is not dominant.} \end{array} \right. \tag{4.26}$$

*Proof.* Consider the $G_j^{(i)}$-product which contains only such $G_j^{(i)}$ factors that satisfy $(\bar{n}-i)\bar{\lambda} - j \ge 0$. Then, (4.25) tells us that

$$\begin{aligned} & \|\ \hat{\gamma}\ \{G_{j_0}^{(0)}/\beta\}^{e_0}\{G_{j_1}^{(i_1)}/\beta\}^{e_1} \cdots \{G_{j_\kappa}^{(i_\kappa)}/\beta\}^{e_\kappa}\| \\ = \ & O(\hat{\gamma})\, O(D^{(1-0)\bar{\lambda}-j_0})^{e_0}\, O(D^{(1-i_1)\bar{\lambda}-j_1})^{e_1} \cdots O(D^{(1-i_\kappa)\bar{\lambda}-j_\kappa})^{e_\kappa} \\ = \ & O(\hat{\gamma})\, O(D^{(\Sigma e_r)\bar{\lambda}-(\Sigma i_r e_r)\bar{\lambda}-(\Sigma j_r e_r)}) = O(\hat{\gamma})\, O(D^{\bar{\lambda}-k}). \end{aligned}$$

Here, the summations are over all the $G_{j_r}^{(i_r)}$ factors in the product, hence $\Sigma_{r=1}^{\kappa} e_r = e$ by (3.11), $\Sigma_{r=1}^{\kappa} i_r e_r = e-1$ by (3.13), and $\Sigma_{r=1}^{\kappa} j_r e_r = k$ by (3.12).

Now, consider the largest $(1/\beta)$-power terms in **3.3** $\sim$ **3.5**. Each of the terms contains only such $G_j^{(i)}$ factors that satisfy $(\bar{n}-i)\bar{\lambda}-j \geq 0$. Furthermore, the coefficient of the term is not zero, and we see $\hat{\gamma} = O(D^0)$. Therefore, we see $\|G_j^{(i)}\text{-product}\| = O(D^{\bar{\lambda}-k})$ for such a term. On the other hand, many $G_j^{(i)}$-products contain such $G_j^{(i)}$ factors that satisfy $(\bar{n}-i)\bar{\lambda}-j < 0$. For such $G_j^{(i)}$-products, we have $\|G_j^{(i)}\text{-product}\| = o(D^{\bar{\lambda}-k})$. □

**Theorem 3** *Let the expansion point be far from the origin:* $D = (|s_1|^2 + \cdots + |s_\ell|^2)^{1/2} \gg 1$. *Choose the lowest order solution* $\alpha$, $|\alpha| \gg 1$, *to be separated from other solutions of* $F(x,s)=0$ *by* $O(|\alpha|)$ *or more. Then, in the computation of* $y_k(v)$ *for any* $k$, *there occurs no large cancellation errors of magnitude* $O(D^{ck})$, $c>0$, *although a cancellation may happen the amount of which is independent of* $k$.

*Proof.* Let $\tilde{\chi}(u)$ be a solution of $F_{\mathcal{S}_\iota}(x,u)=0$, such that $\tilde{\chi}(s)=\alpha$ and the Taylor expansion of $\tilde{\chi}(s+v)$ gives the dominant terms of the power series solution $\chi^{(\infty)}(v)$. Then, $\tilde{\chi}(u)$ is a homogeneous algebraic function such that $\tilde{\chi}(tu) = t^{\bar{\lambda}}\tilde{\chi}(u)$. Hence, expanding $\tilde{\chi}(s+v)$ into Taylor series in $v_1/s_1,\ldots,v_\ell/s_\ell$, as in Lemma 1 in **2.4**, we see that $\|y_k(v)\| = O(D^{\bar{\lambda}-k})$. On the other hand, Prop. 3 tells us that $\|\text{dominant } G_j^{(i)}\text{-product}\| = O(D^{\bar{\lambda}-k})$ in $y_k(v)$. Therefore, there occurs no large cancellation error of magnitude $O(D^{ck})$, $c>0$, in the computation of $y_k(v)$. □

**Remark 5** Eq. (4.26) shows that the magnitude of coefficients of $y_k$ decreases very rapidly as $k$ increases, which may cause underflow of numbers in floating-point arithmetic. We had better make a suitable scale transformation to avoid the underflow. □

# 5 Miscellaneous remarks on errors

In this section, we consider to overcome large cancellation errors which may occur near a singular point, analyze the case of "close" roots which was avoided in **4.4**, and the accumulation of rounding errors.

## 5.1 Error-safe expansion near a singular point

In this subsection, we propose an error-safe expansion method near a singular point at which Newton's method may cause extremely large cancellation errors. Our idea is to utilize the expansion method at the singular point. According to **2.3**, the expansion at a singular point necessitates us in general to introduce an algebraic function $\theta(u)$, hence the method cannot directly be used for calculating the power series root $\chi^{(k)}(v)$. Fortunately, large cancellation errors occur only when $\hat{\chi}^{(k)}(u)$, the root expanded at the singular point, is integral w.r.t. some variable. Furthermore, if $\hat{\chi}^{(k)}(u)$ is integral w.r.t. $x_l$ but non-integral w.r.t. $x_{l'}$ then the terms containing $x_l$ do not give dominant terms in $y_k(v)$. Therefore, so long as considering the cancellation errors in dominant terms, we have only to consider the case that $\hat{\chi}^{(k)}(u)$ is integral w.r.t. every variable. Then, we can calculate $\hat{\chi}^{(k)}(u)$ just as calculating $\chi^{(k)}(v)$, without introducing $\theta(u)$.

For convenience of explanation, we confine ourselves to the case of $\ell = 1$ (the method itself is applicable to the case of $\ell \geq 2$). Let the expansion point be $u_1 = s_1$ which is very close to a singular point located at $u_1 = \hat{s}_1$ :

$$|s_1 - \hat{s}_1| \ll 1. \tag{5.1}$$

Suppose that we want to calculate the power series root to order $k$ with numerical accuracy $\epsilon_M$. Using the method described in **2.3**, we first calculate the power series at root the singular point, up to order $k' > k$, where the value of $k'$ is specified below; note that $\hat{\chi}^{(k')}(u_1)$ is a polynomial in $u_1 - \hat{s}_1$ because we assumed it to be integral.

$$\hat{\chi}^{(k')}(u_1) \;=\; \hat{\alpha} + \hat{c}_1(u_1 - \hat{s}_1) + \cdots + \hat{c}_{k'}(u_1 - \hat{s}_1)^{k'}. \tag{5.2}$$

Here, the numbers $\hat{c}_i, \ldots, \hat{c}_{k'}$ are calculated to accuracy $\epsilon_M$. Then, we substitute $s_1 + v_1$ for $u_1$ in $\hat{\chi}^{(k')}(u_1)$, obtaining $\chi^{(k)}(v_1)$ as follows.

$$\chi^{(k)}(v_1) \;=\; \alpha + c_1 v_1 + \cdots + c_k v_1^k, \tag{5.3}$$

where $\alpha, c_1, \ldots, c_k$ are calculated as

$$\begin{array}{rcl} \alpha &=& \hat{\alpha} + \hat{c}_1(s_1 - \hat{s}_1) + \cdots + \hat{c}_{k'}(s_1 - \hat{s}_1)^{k'}, \\ c_1 &=& \hat{c}_1 + 2\hat{c}_2(s_1 - \hat{s}_1) + \cdots + k'\hat{c}_{k'}(s_1 - \hat{s}_1)^{k'-1}, \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ c_k &=& \hat{c}_k + {}_{k+1}C_1\, \hat{c}_{k+1}(s_1 - \hat{s}_1) + \cdots + {}_{k'}C_{k'-k}\, \hat{c}_{k'}(s_1 - \hat{s}_1)^{k'-k}, \end{array} \tag{5.4}$$

with ${}_iC_j$'s the binomial coefficients. We determine $k'$ to satisfy

$$\left\{ \begin{array}{l} \hat{c}_{k'}(s_1 - \hat{s}_1)^{k'} \text{ is a dominant term in } \hat{\chi}^{(k')}(u_1), \\ |\, {}_{k'}C_{k'-i}\, \hat{c}_{k'}(s_1 - \hat{s}_1)^{k'-i}| < \epsilon_M \quad (i = 0, 1, \ldots, k). \end{array} \right. \tag{5.5}$$

The computation of $\hat{\chi}^{(k')}(u_1)$ causes no large error usually, as we will see from an example. Therefore, if we want to calculate $\chi^{(k)}(v_1)$ to accuracy $10^{-15}$ with $|s_1 - \hat{s}_1| = 10^{-4}$, for example, we have to calculate $\hat{\chi}^{(k+3)}(u_1)$ to accuracy $10^{-15}$.

**Example 7** Error-safe expansion of $F(x, u)$ given in Example 1.

$$\begin{array}{rcl} F(x,u) &=& x^6 - 3(u-1)x^4 - 2ux^3 + 3(u^2 - 2u + 1)x^2 \\ && \qquad - 6(u^2 - u)x - u^3 + 4u^2 - 3u + 1. \end{array}$$

$F(x, u)$ has a singular point at $u = \hat{s} = 0.516926102175\cdots$, and $F(x, \hat{s})$ has a double root $\hat{\alpha} = -0.401278746768\cdots$. In order to apply the method given in **2.3** directly to our problem, we shift the origin to the point $(x, u) = (\hat{\alpha}, \hat{s})$ :

$$\begin{array}{rcl} \hat{F}(\hat{x}, \hat{u}) &\stackrel{\text{def}}{=}& F(\hat{x} - 0.40127874676866, \hat{u} + 0.51692610217531) \\ &=& \hat{x}^6 - 2.407\cdots \hat{x}^5 + (3.864\cdots - 3\hat{u})\hat{x}^4 \\ && + (4.652\cdots + 2.815\cdots \hat{u})\hat{x}^3 + (3.733\cdots - 3.389\cdots \hat{u} + 3\hat{u}^2)\hat{x}^2 \\ && + (1.932\cdots \hat{u} - 8.407\cdots \hat{u}^2)\hat{x} + (5.339\cdots \hat{u}^2 - \hat{u}^3). \end{array}$$

Here and throughout this example, we cutoff the terms of magnitudes $O(10^{-13})$ or less, which discards all the fully erroneous terms. $\hat{F}(\hat{x},\hat{u})$ contains factors of degrees 2 and 4 which are singular and non-singular at the origin, respectively. The Newton polynomial for the singular factor is $\hat{F}_{\rm New}(\hat{x},\hat{u}) = 3.733\cdots\hat{x}^2 + 1.932\cdots\hat{u}\hat{x} + 5.339\cdots\hat{u}^2$, and the solution of $\hat{F}_{\rm New}(\hat{x},\hat{u}) = 0$ gives the first order solution $\hat{\chi}^{(1)}(\hat{u})$ of $\hat{F}(\hat{x},\hat{u}) = 0$: $\hat{\chi}^{(1)}(\hat{u}) = (-0.25875958225623{+}1.1675727920436\,i)\hat{u}$. (We have chosen one of the complex conjugate solutions. Note that this solution can be obtained by solving $\hat{F}_{\rm New}(\hat{x},1) = 0$.) Putting $\hat{\chi}^{(k)} = \hat{\chi}^{(1)} + \hat{d}^{(2)} + \hat{d}^{(3)} + \cdots$, with $\mathrm{ord}(d^{(i)}) = i$, and substituting this for $\hat{x}$ in $\hat{F}(\hat{x},\hat{u})$, we can determine $\hat{d}^{(2)}, \hat{d}^{(3)}, \ldots$ successively. For example,

$$
\begin{array}{rcl}
\hat{\chi}^{(5)} & = & (-0.25875958225623 + 1.1675727920436\,i)\;\hat{u} \\
 & + & (\;\;0.1668578810829\quad + 0.083290774208661\,i)\;\hat{u}^2 \\
 & + & (-0.17932703036999 + 0.6959452980097\,i)\;\hat{u}^3 \\
 & + & (\;\;0.23127358645056 + 0.09797677528285\,i)\;\hat{u}^4 \\
 & + & (-0.32809453682074 + 1.2907084531487\,i)\;\hat{u}^5.
\end{array}
$$

Finally, substituting $v + (0.517 - \hat{s})$ for $u$ in $\hat{\chi}^{(5)}$ and adding $\hat{\alpha}$ to the result, we obtain $\chi^{(3)}$ (below, the underlined figures are correct while the others are wrong).

$$
\begin{array}{rcl}
\chi^{(3)} & = & (-\underline{0.4012978676277}9 - \underline{0.00008628154462}1635\,i) \\
 & + & (-\underline{0.2587349243248}2 - \underline{1.1675851134593}\,i)\;v \\
 & + & (\;\;\underline{0.1668181330269}7 - \underline{0.08344506395498}9\,i)\;v^2 \\
 & + & (-\underline{0.1792586858270}8 - \underline{0.6959743295761}\,i)\;v^3.
\end{array}
$$

We comment that there occurs no large cancellation error in the calculation of $\hat{\chi}^{(5)}$. Since $|0.517 - \hat{s}| \approx 10^{-4}$, the $\hat{u}^6$-term in $\hat{\chi}^{(6)}$ gives a correction of magnitude $O(10^{-4(6-i)})$ to the $v^i$-term in $\chi^{(3)}$, hence all the terms in $\chi^{(3)}$ are accurate to $O(10^{-12})$. $\Box$

## 5.2 Expansion at a distant point near a singular point

In **4.4**, we have investigated the expansion at a distant point, assuming that there is no singular point near the expansion point. In this subsection, we consider the case that there is a singular point near the expansion point. We use the same notations as in **4.4**, and assume that $(\hat{\alpha},\hat{s})$ is a singular point of $F(x,u)$, of multiplicity $m$, in the sense of algebraic geometry. We also assume that the expansion point is close to the singular point and there is no other singular point near the expansion point. We put

$$
D' \stackrel{\rm def}{=} (|s_1 - \hat{s}_1|^2 + \cdots + |s_\ell - \hat{s}_\ell|^2)^{1/2} \;\ll\; D. \tag{5.6}
$$

Let $\hat{\chi}(u)$ be a root w.r.t. $x$, of $F(x,u)$, satisfying $\hat{\chi}(\hat{s}) = \hat{\alpha}$. Since the singular point closest to the expansion point is located at $(u) = (\hat{s})$, $\chi^{(k)}(v)$ can be obtained by expanding $\hat{\chi}(u)$ into Taylor series in $v_1/(s_1 - \hat{s}_1), \ldots, v_\ell/(s_\ell - \hat{s}_\ell)$. Therefore, we have

$$
\lim_{k\to\infty} \sup\, \|y_{k+1}(v)\| / \|y_k(v)\| \;=\; 1/D'. \tag{5.7}
$$

Now, let us first order estimate $\alpha$ by remembering the arguments in **4.1** and **4.4**. By assumption, $F(x,\hat{s})$ has $m$ multiple root $\hat{\alpha}$ of magnitude $O(D^{\bar{\lambda}})$, and the root $\hat{\alpha}$ will split into $m$ "close" roots $\alpha \stackrel{\rm def}{=} \alpha_1, \ldots, \alpha_m$ in $F(x,s) = G(x,0)$. $F(x,s)$ and $F(x,\hat{s})$ are related with each other by

$$F(x,s) \;=\; F(x,\hat{s}) + \sum_{j=1}^{\tau} \frac{1}{j!} \Big[ (s_1 - \hat{s}_1)\frac{\partial}{\partial u_1} + \cdots + (s_\ell - \hat{s}_\ell)\frac{\partial}{\partial u_\ell} \Big]^j F(x,u)\Big|_{(u)=(\hat{s})}.$$

For each $l$ $(1 \le l \le \ell)$, substitution of $(\hat{s})$ for $(u)$ in $\frac{\partial}{\partial x_l} F(x,u)$ causes usually no large cancellation. Hence, the assumption $D = \|s\| \gg \|s - \hat{s}\| = D'$ tells us that

$$\frac{|f_i(s) - f_i(\hat{s})|}{|f_i(s)|} = O(D'/D) \quad (i = 0, 1, \ldots, n). \tag{5.8}$$

Therefore, $|\alpha_i - \hat{\alpha}|$ is order bounded as

$$|\alpha_i - \hat{\alpha}| \le O(D^{\bar{\lambda}}) \cdot O([D'/D]^{1/m}) \quad (1 \le i \le m). \tag{5.9}$$

Although the root $\hat{\chi}(u)$ behaves variously around the singular point, we classify the situation into the following three cases which satisfy (5.9).

$$\frac{|\alpha - \hat{\alpha}|}{|\hat{\alpha}|} \;\stackrel{\rm def}{=}\; O([D'/D]^{\eta_\alpha}) \;=\; \begin{cases} \text{Case 1:} & O([D'/D]^{\eta_1}), \quad 1/m \le \eta_1 < 1, \\ \text{Case 2:} & O([D'/D]^{\eta_2}), \quad \eta_2 = 1, \\ \text{Case 3:} & O([D'/D]^{\eta_3}), \quad \eta_3 > 1. \end{cases} \tag{5.10}$$

Remembering that $F(x,s) \simeq F_{\mathcal{S}_\iota}(x,s)$ and that $F_{\mathcal{S}_\iota}(x,tu)$ is homogeneous w.r.t. $x$ and $t^{\bar{\lambda}}$, we obtain the following order estimation of $\beta$ directly from (5.10).

$$\beta \;=\; O(D^{(\bar{n}-1)\bar{\lambda}+\bar{\tau}}) \cdot O([D'/D]^{(m-1)\eta_\alpha}). \tag{5.11}$$

The argument in **4.1** tells us that the $\alpha$-substitution for $\|G_j^{(i)}(x,v)\|$, with $i+j<m$, causes the cancellation of magnitude $O(\max\{|\alpha - \hat{\alpha}|/|\hat{\alpha}|,\, \|s - \hat{s}\|/\|\hat{s}\|\}^{m-(i+j)})$. Hence, we define a number $\eta$ as

$$\max\Big\{ \frac{|\alpha - \hat{\alpha}|}{|\hat{\alpha}|}, \frac{\|s - \hat{s}\|}{\|\hat{s}\|} \Big\} \;\stackrel{\rm def}{=}\; O([D'/D]^{\eta}) \;=\; \begin{cases} \text{Case 1:} & O([D'/D]^{\eta_1}), \\ \text{Case 2:} & O([D'/D]), \\ \text{Case 3:} & O([D'/D]). \end{cases} \tag{5.12}$$

Then, we obtain the following order estimation of $\|G_j^{(i)}(\alpha,v)\|$, where we consider only such $i$ and $j$ that satisfy $(\bar{n}-i)\bar{\lambda} - j \ge 0$, as in **4.4**.

$$\|G_0^{(i>1)}(\alpha)\| \;=\; O(D^{(\bar{n}-i)\bar{\lambda}-0+\bar{\tau}}) \cdot O([D'/D]^{\max\{0,\,(m-i)\eta_\alpha\}}), \tag{5.13}$$

$$\|G_{j>0}^{(i)}(\alpha,v)\| \;=\; O(D^{(\bar{n}-i)\bar{\lambda}-j+\bar{\tau}}) \cdot O([D'/D]^{\max\{0,\,(m-i-j)\eta\}}). \tag{5.14}$$

**Proposition 4** *The dominant $G_j^{(i)}$-product in* (3.10) *is order estimated as follows.*

$$\|\text{dominant } G_j^{(i)}\text{-product}\| \;=\; \begin{cases} \text{Case 1:} & O(D^{\bar{\lambda}-k}) \cdot O([D'/D]^{\eta_1(1-k)}), \\ \text{Case 2:} & O(D^{\bar{\lambda}-k}) \cdot O([D'/D]^{1-k}), \\ \text{Case 3:} & O(D^{\bar{\lambda}-k}) \cdot O([D'/D]^{(1-k)-(\eta_3-1)(mk-m+1)}). \end{cases} \tag{5.15}$$

*Proof.* We consider only the dominant $G_j^{(i)}$-products which satisfy $(\bar{n}-i)\bar{\lambda}-j \geq 0$, as in **4.4**. Since each $G_j^{(i)}(\alpha, v)$ is accompanied with the factor $1/\beta$, we consider $G_j^{(i)}(\alpha, v)/\beta$. Since $D'/D \ll 1$ and $\max\{0, a\} \geq a$, (5.11) and (5.13), (5.14) give

$$\begin{aligned}
\|G_0^{(i>1)}(\alpha)/\beta\| &\leq O(D^{(1-i)\bar{\lambda}-0}) \cdot O([D'/D]^{(1-i)\eta_\alpha}), \\
\|G_{j>0}^{(i)}(\alpha, v)/\beta\| &\leq O(D^{(1-i)\bar{\lambda}-j}) \cdot O([D'/D]^{(1-m)\eta_\alpha+(m-i-j)\eta}).
\end{aligned}$$

Note that there are some $G_0^{(i>1)}(\alpha)$'s and $G_{j>0}^{(i)}(\alpha, v)$'s for which the "=" holds in the above two relations. Therefore, the dominant $G_j^{(i)}$-product can be order estimated by the above r.h.s. expressions. We rewrite the last factors in the above expressions as

$$\begin{aligned}
O([D'/D]^{(1-i)\eta_\alpha}) &= O([D'/D]^{(1-i-0)\eta+(m-1)(\eta-\eta_\alpha)-(m-i)(\eta-\eta_\alpha)}), \\
O([D'/D]^{(1-m)\eta_\alpha+(m-i-j)\eta}) &= O([D'/D]^{(1-i-j)\eta+(m-1)(\eta-\eta_\alpha)}).
\end{aligned}$$

We substitute the above order estimations for $G_j^{(i)}(\alpha, v)$ in (3.10), and perform a similar calculation as in the proof of Prop. 3 (below, $i'_1, \ldots, i'_\mu > 1$ and $j_1, \ldots, j_\kappa \geq 1$) :

$$\begin{aligned}
&\|\text{dominant-}\{G_0^{(i'_1)}/\beta\}^{e'_1} \cdots \{G_0^{(i'_\mu)}/\beta\}^{e'_\mu} \cdot \{G_{j_1}^{(i_1)}/\beta\}^{e_1} \cdots \{G_{j_\kappa}^{(i_\kappa)}/\beta\}^{e_\kappa}\| \\
&= O(D^{\bar{\lambda}-k}) \cdot O([D'/D]^{\eta(1-k)+(m-1)(\eta-\eta_\alpha)e}) \cdot O([D'/D]^{(\eta_\alpha-\eta)\,\Sigma_{r=1}^{\mu}(m-i'_r)e'_r}). \quad (5.16)
\end{aligned}$$

Here, we have used the relations $\Sigma_{r=1}^{\mu} e'_r + \Sigma_{r=1}^{\kappa} e_r = e$ by (3.11), $\Sigma_{r=1}^{\mu} i'_r e'_r + \Sigma_{r=1}^{\kappa} i_r e_r = e-1$ by (3.13), and $\Sigma_{r=1}^{\mu} j'_r e'_r + \Sigma_{r=1}^{\kappa} j_r e_r = k$ by (3.12).

In the Cases 1 and 2, we need not to calculate any more because $\eta_\alpha = \eta$. Let us consider (5.16) in the Case 3. Since $\eta_\alpha - \eta > 0$ in the Case 3, one may think that the above expression becomes the largest when $m - i'_r$ becomes the smallest. However, this is wrong because the value of $e$ also varies. Suppose $i_1, \ldots, i_\kappa$ are fixed in (5.16), then the value of $e$ increases by $e'_i$ if $i'_r$ is increased by 1. Therefore, the r.h.s. of (5.16) becomes the largest when $i'_r$ becomes the smallest, or $i'_r = 2$. Lemma 3 shows that $e$ becomes its largest value $2k-1$ when $\mu = 1$, $i'_1 = 2$ and $e'_1 = k-1$. Therefore, we obtain (5.15). □

**Remark 6** Proposition 4 shows that, in the Case 1, $\|y_k\|$ decreases more slowly than $[1/D']^k$ as $k$ increases. This contradicts (5.7), hence the authors doubt if the Case 1 occurs actually. In the following, we discard the Case 1. □

Proposition 4 and (5.2) give the following theorem directly.

**Theorem 4** *Let the expansion point $(s)$ be far from the origin*: $(|s_1|^2+\cdots+|s_\ell|^2)^{1/2} \stackrel{\text{def}}{=} D \gg 1$. *Assume that there is a singular point at $(\hat{s})$ which is close to the expansion point and there is no other singular point around it*: $(|s_1-\hat{s}_1|)^2+\cdots+|s_\ell-\hat{s}_\ell|^2)^{1/2} \stackrel{\text{def}}{=} D' \ll D$. *Let $\hat{\alpha}$ be a multiple root of $F(x, \hat{s})$, of multiplicity $m$, and $\hat{\alpha}$ be splitted into $m$ close roots $\alpha \stackrel{\text{def}}{=} \alpha_1, \ldots, \alpha_m$, of $F(x, s)$, satisfying* (5.10). *Then, during the computation of $y_k(v)$, there occurs no large cancellation error (except for the cancellation by $\alpha$-substitution) in the Case 2, while there occur large cancellation errors of magnitude $O([D/D']^{(\eta_3-1)mk})$ in the Case 3.*

**Example 8** Expansion at a distant point near a singular point.

$$\begin{aligned} F(x,u) &= x^6 - 3(u-1)x^4 - 2ux^3 + 3(u^2-2u+1)x^2 \\ &\quad - 6(u^2-u)x - (3/4)u^3 + 4u^2 - 3u + 1. \end{aligned} \tag{5.17}$$

This is the same as $F(x,u)$ in Example 1 except for the coefficient of $u^3$-term. $F(x,u)$ has two distant singular points at $\hat{s}_1 = 1270.84\cdots$ and $\hat{s}_2 = 753.57\cdots$. In fact, $F(x,\hat{s}_1)$ has a double root at $\hat{\alpha} = 39.644\cdots$.

We choose the expansion point at $u = s = 1260$ which is close to the singular point $\hat{s}_1 \stackrel{\text{def}}{=} \hat{s}$. We note that $D' \approx 10.8$ and $D'/D \approx 0.0086$. Then, we have

$$\begin{aligned} G(x,v) &= x^6 - (3v+3777)x^4 - (2v+2520)x^3 + (3v^2+7554v+4755243)x^2 \\ &\quad - (6v^2+15114v+9518040)x - 3v^3/4 - 2831v^2 - 3562023v - 1493935379. \end{aligned}$$

Calculating the roots of $G(x,0)$, we see that the double root $\hat{\alpha}$ of $F(x,\hat{s})$ splits into two "close" roots: $\alpha_1 = 38.405\cdots$ and $\alpha_2 = 40.096\cdots$. Note that $\bar{\lambda} = 1/2$ and $\alpha_i = O(D^{\bar{\lambda}})$, $|\alpha_i - \hat{\alpha}|/|\hat{\alpha}| = O(D'/D)$. We choose $\alpha_2$ as $\alpha$: $\alpha \stackrel{\text{def}}{=} \alpha_2$. Then, we obtain $\beta = 7582806.25\cdots$. Calculating the power series root up to order 8, we find

$$\begin{aligned} \chi^{(8)} &= 40.09\cdots - 0.01232\cdots v - 0.0006800\cdots v^2 \\ &\quad - 3.119\cdots\times 10^{-5}v^3 - 1.798\cdots\times 10^{-6}v^4 - 1.160\cdots\times 10^{-7}v^5 \\ &\quad - 8.029\cdots\times 10^{-9}v^6 - 5.818\cdots\times 10^{-10}v^7 - 4.359\cdots\times 10^{-11}v^8. \end{aligned}$$

As we have mentioned above, the coefficient of $v^k$-term decreases by $O(1/D')$ as $k$ increases by 1. The computation of $\chi^{(8)}$ causes errors of magnitude $O(D/D')$ in $\beta$ and $c_1$ due to the $\alpha$-substitution, where $c_k$ is the coefficient of $v^k$-term of $\chi^{(8)}$. However, the computation causes no additional remarkable errors in $c_2 \sim c_8$, as predicted by theory. □

## 5.3 On accumulation of rounding errors

Accurate estimation of accumulation of rounding errors in formula (3.7) is pretty difficult because of the following two reasons. First, $y_k(v)$ is composed of quite many $G_j^{(i)}$-products. Second, the accumulation of rounding errors varies largely by situation. For example, let $a = \bar{a} + \epsilon_a$ and $b = \bar{b} + \epsilon_b$ be two floating-point numbers, where $\bar{a}$ and $\bar{b}$ are correct values and $\epsilon_a$ and $\epsilon_b$ are small errors. Then, we have

$$\begin{aligned} |\text{relative error in } a+b| &\simeq |\epsilon_a + \epsilon_b|/|\bar{a}+\bar{b}| \le (|\epsilon_a| + |\epsilon_b|)/|\bar{a}+\bar{b}|, \\ |\text{relative error in } a\times b| &\simeq |\epsilon_a/\bar{a} + \epsilon_b/\bar{b}| \le (|\epsilon_a/\bar{a}| + |\epsilon_b/\bar{b}|), \end{aligned}$$

which show that accumulated error varies largely with the values of $\bar{a}$, $\bar{b}$, $\epsilon_a$, and $\epsilon_b$. Only one exceptional operation is the exponentiation, which makes the accumulated error almost maximum: $(\bar{a}+\epsilon_a)^k \simeq \bar{a}^k + k\epsilon_a\bar{a}^{k-1}$.

Therefore, among the $G_j^{(i)}$-products in $y_k(v)$, one which contains the largest power of $1/\beta$ and the smallest number of $G_j^{(i)}$ factors, or $G_0^{(2)}(\alpha)^{k-1}G_1(\alpha,v)^k/\beta^{2k-1}$, will accumulate the rounding errors maximally. Note that this term is usually a dominant term in $y_k(v)$. Hence, we estimate the accumulation of rounding errors roughly by estimating the errors in this term.

Let $\epsilon_0$ and $E_1(v)$ be the error terms in $G_0^{(2)}(\alpha)/\beta$ and $G_1(\alpha,v)/\beta$, respectively, with $|\epsilon_0| \ll |G_0^{(2)}(\alpha)/\beta$ and $\|E_1(v)\| \ll \|G_1(\alpha,v)/\beta\|$. Then, we have

$$\begin{aligned} &[\text{error term in } \{G_0^{(2)}(\alpha)/\beta\ +\epsilon_0\}^{k-1}\{G_1(\alpha,v)/\beta + E_1(v)\}^k] \\ &\simeq\ \{\,(k-1)\,\epsilon_0\, G_0^{(2)}(\alpha)/\beta + k\, E_1(v)\, G_1(\alpha,v)/\beta\,\}\{G_0^{(2)}(\alpha)/\beta\}^{k-2}\{G_1(\alpha,v)/\beta\}^{k-1}. \end{aligned} \tag{5.18}$$

This shows that the rounding errors in $y_k(v)$ will increase in proportion to $k$.

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