

An example of the Application of Computer Algebra approach to Solving Engineering Problems

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Abstract: - In this paper, we present an example of the application of the computer algebra approach to solving engineering problems from the point of view of showing the capabilities of symbolic computations as a tool for real technical problems. Namely we consider the problem of power loop systems design and show it based on one practical example, that the problem may be reduced to solving a system of non-linear equations and that the computational software program Mathematica may be applied to solve the system symbolically and the corresponding solutions yield parameters of the circuit corresponding to the initial design target.

Key-Words: - symbolic computations, computer algebra, numerical computations, circuit design

1 Introduction

Recently, some studies of the application of the computer algebra approach to solving engineering problems are reported. In the previous work focused on the computer algebra approach to the integrated circuits, the estimation of signal delay calculation or thermal analytical model of signal interconnect for digital integrated circuits by using computer algebra have been discussed[9][10]. Moreover, there has been intensive research on symbolic analysis for design of analog integrated circuits because lack of automation tools can be found [2][3]. Especially, regarding the algebraic capabilities of modern computer algebra systems, some examples for designing analog circuits were investigated [5][6]. In their previous work, they showed that symbolic method applied to simple circuit design takes advantage of finding solutions for the nonlinear equations.

In our research, we apply to power loop systems design which is more complex than conventional work and particularly important in the modern integrated circuit design. As first, we modelled an amplifier circuit and a LCR circuit with transfer function. In the next, we modelled an error detection and feedback loop. Furthermore we derived a transfer function of the power supply stabilization loop circuit by composing these functions. Finally, we established a design target used in industrial

design and determined the most suitable circuit parameters to satisfy it. For getting these parameters, we solved a system of nonlinear equations using symbolic algebraic manipulation so-called Mathematica [7]. As a result, we were not able to find a set of constants of circuit elements that is applicable to an industrial circuit design with the conventional numerical method. But we applied symbolic algebra method to those nonlinear equations, then we obtained plural solutions. In addition, we found that each of the solutions had important sense as a circuit parameter that constructed whole system. Now we discuss how to symbolically solve nonlinear equations and what solution can be obtained. In this way, we propose our useful symbolic method for a power supply stabilization loop circuit.

2 The General Approach

One of the principal issues in analog circuit design is to find the most suitable circuit parameter with resulting transfer function to satisfy previously given specifications. In general, the network functions for analog circuits are rational functions (ratio of two polynomials) in the complex frequency variable s [3][4]:

$$H(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m} \quad (1)$$

, where the numbers m, n and the coefficients a_i ($0 \leq i \leq n$), b_j ($0 \leq j \leq m$) are determined by the given specifications and both the numerator and denominator polynomial are symbolic polynomial functions that are represented by a symbol instead of a numerical value. The following example leads us to the various definition of the symbolization. The circuit elements represented by symbols depending on whether all, or some of them are distinguished in accordance with given or determined parameters [1][2]:

Fully symbolic network function:

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + RCs + LCs^2} \quad (2)$$

Partially symbolic and partially numerical network function:

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + 5.0 - 11Rs + 3.0 - 11s^2} \quad (3)$$

Now, in order to determine a target analog circuit, each element shown earlier is described by the following network function

$$T(s) = \frac{\sum_{\mu=0}^n A_{\mu}(p_1, \dots, p_k) s^{\mu}}{\sum_{\tau=0}^m B_{\tau}(p_1, \dots, p_k) s^{\tau}} \quad (4)$$

where the terms A_{μ} and B_{τ} are polynomials in p_1, \dots, p_k satisfies

$$H(s) = T(s) \quad (5)$$

To solve equation (5) such as nonlinear simultaneous equations, we tried to apply symbolic and numerical analysis and confirmed suitable solution for designer by not numerical but symbolic analysis.

3 Modeling of the system

Consider a power loop system as depicted in Fig.1, which composed from three sub blocks -

block1, block2 and block3 - surrounded with a dotted line.

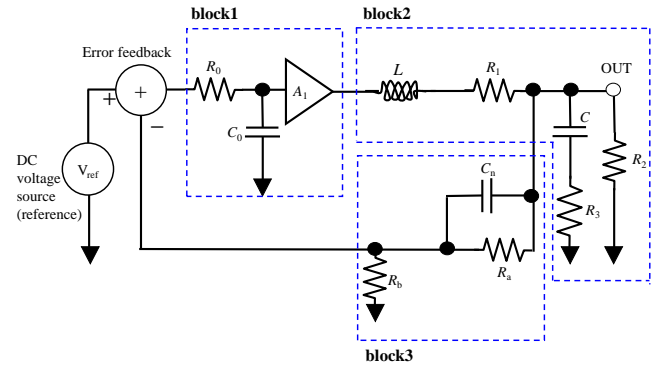
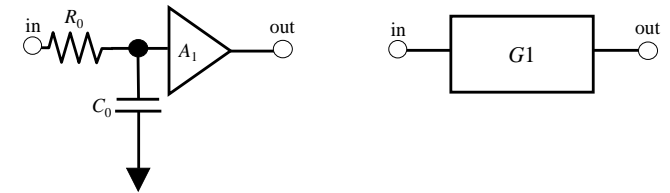


Fig.1: A power loop system which has three blocks

3.1 Modeling of block1

Block1 showed in Fig. 1 is an amplifier circuit. Now we describe its model and transfer function in Fig.2.



(A)An amplifier circuit model (B)Transfer function of (A)

Fig.2: Model and transfer function of Block1

We see that the symbolic transfer function $G1(s)$ of this circuit is given by :

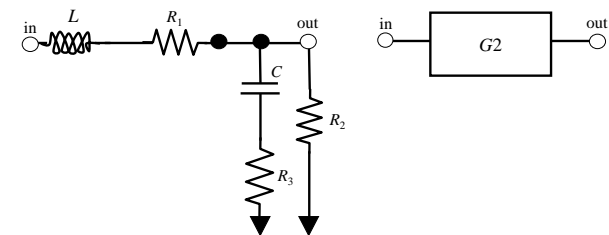
$$\omega_1 = 1/(R_0 C_0) = 2\pi f_1$$

$$G1(s) = \left(\frac{1/sC_0}{R_0 + 1/sC_0} \right) A_1 = \left(\frac{1/sR_0C_0}{1 + 1/sR_0C_0} \right) A_1 = \left(\frac{1/R_0C_0}{s + 1/R_0C_0} \right) A_1 = \left(\frac{\omega_1}{s + \omega_1} \right) A_1 \quad (6)$$

Note that f_1 is frequency, and we are to set a value to f_1 later.

3.2 Modeling of block2

Block2 showed in Fig.1 is a familiar LCR circuit. In the same way as previous, we describe its model and transfer function in Fig.3.



(A)An amplifier circuit model (B)Transfer function of (A)

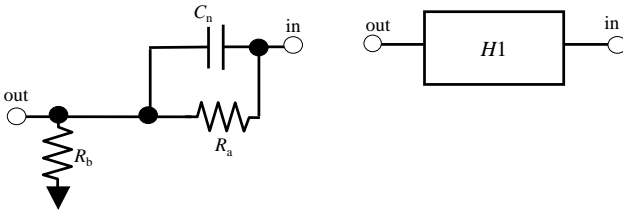
Fig.2: Model and transfer function of Block2

We got that the symbolic transfer function $G2(s)$ of this circuit is given by :

$$G2(s) = \frac{(R_3 + \frac{1}{sC})R_2}{\left\{ R_1 + sL + \left(\frac{R_2 R_3}{R_3 + R_2 + \frac{1}{sC}} \right) + \frac{R_2}{sC(R_3 + R_2 + \frac{1}{sC})} \right\} (R_3 + R_2 + \frac{1}{sC})} \quad (7)$$

3.3 Modeling of block2

Finally, block3 showed in Fig.31 is feedback circuit as we know.



(A)An amplifier circuit model (B)Transfer function of (A)
Fig.3: Model and transfer function of Block2

We solved that the symbolic transfer function $H1(s)$ of this circuit is :

$$H1(s) = \frac{R_b}{R_b + \left(\frac{R_a \frac{1}{sC_n}}{R_a + \frac{1}{sC_n}} \right)} = \frac{R_b}{R_b + \left(\frac{R_a}{1 + sR_a C_n} \right)} \quad (8)$$

3.4 Modeling of the fully system

Now we were to obtain transfer function of the fully system which constructed from block1, block2 and block3 showed in Fig. 1. As we illustrated in Fig.5, the whole system has three series of transfer function $G1$, $G2$ and $H1$ as shown previously.

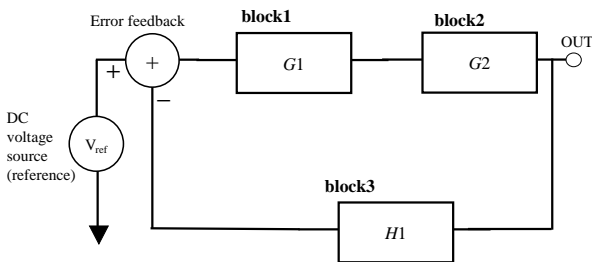


Fig.5: Model with transfer function of our system

In order to get the symbolic transfer function of fully systems, with symbolic and algebraic manipulation called Mathematica (Version 8.0.0.0)[7], first we connected $G1$ and $G2$ by using “SystemModelSeriesConnect” command, then

connected it and $H1$ by using “SystemModelFeedbackConnect” command. As a result, we could get that the systems symbolic transfer function is given by

$$T_{sys} = \frac{U_2 s^2 + U_1 s + U_0}{D_4 s^4 + D_3 s^3 + D_2 s^2 + D_1 s + D_0} \quad (9)$$

where,

$$\begin{aligned} U_0 &= A_1 \omega_1 R_2 R_a + A_1 \omega_1 R_2 R_b, \\ U_1 &= A_1 C \omega_1 R_2 R_3 R_a + A_1 C \omega_1 R_2 R_3 R_b + A_1 C_n \omega_1 R_2 R_a R_b, \\ U_2 &= A_1 C C_n \omega_1 R_2 R_3 R_a R_b, \\ D_0 &= \omega_1 R_1 R_a + \omega_1 R_2 R_a + \omega_1 R_1 R_b + \omega_1 R_2 R_b + A_1 \omega_1 R_2 R_b, \\ D_1 &= L \omega_1 R_a + R_1 R_a + R_2 R_a + C \omega_1 R_1 R_2 R_a + C \omega_1 R_1 R_3 R_a + \\ &\quad C \omega_1 R_2 R_3 R_a + L \omega_1 R_b + R_1 R_b + R_2 R_b + C \omega_1 R_1 R_2 R_b + \\ &\quad C \omega_1 R_1 R_3 R_b + C \omega_1 R_2 R_3 R_b + A_1 C \omega_1 R_2 R_3 R_b + \\ &\quad C_n \omega_1 R_1 R_a R_b + C_n \omega_1 R_2 R_a R_b + A_1 C_n \omega_1 R_2 R_a R_b, \\ D_2 &= L R_a + C L \omega_1 R_2 R_a + C R_1 R_2 R_a + C L \omega_1 R_3 R_a + \\ &\quad C R_1 R_3 R_a + C R_2 R_3 R_a + L R_b + C L \omega_1 R_2 R_b + C R_1 R_2 R_b + \\ &\quad C L \omega_1 R_3 R_b + C R_1 R_3 R_b + C R_2 R_3 R_b + C_n L \omega_1 R_a R_b + \\ &\quad C_n R_1 R_a R_b + C_n R_2 R_a R_b + C C_n \omega_1 R_1 R_2 R_a R_b + \\ &\quad C C_n \omega_1 R_1 R_3 R_a R_b + C C_n \omega_1 R_2 R_3 R_a R_b + \\ &\quad A_1 C C_n \omega_1 R_2 R_3 R_a R_b, \\ D_3 &= C L R_2 R_a + C L R_3 R_a + C L R_2 R_b + C L R_3 R_b + \\ &\quad C_n L R_a R_b + C C_n L \omega_1 R_2 R_a R_b + C C_n R_1 R_2 R_a R_b + \\ &\quad C C_n L \omega_1 R_3 R_a R_b + C C_n R_1 R_3 R_a R_b + C C_n R_2 R_3 R_a R_b, \\ D_4 &= C C_n L R_2 R_a R_b + C C_n L R_3 R_a R_b. \end{aligned} \quad (10)$$

Here we suppose to know some parameters in accordance with some previously given performance specification as design. So we fix $A1=60(\text{dB})$, and $L=10(\mu\text{H})$. Also we suppose $\omega_1 = 2\pi f_1 = 2\pi$ because of $f_1=1(\text{Hz})$ given as before. As a result, equation (10) has seven unknown parameters. As you know, it is difficult to obtain numerical solution with Newton-Raphson methods approximately for nonlinear simultaneous equations.

4 Analysis and results

Now we tried to solve this design task. We set our design target as follows: As a state of stabilization of the power supply loop circuit, phase margin is large enough and gain margin is about 20 (dB) as shown in Fig.6. We set the following transfer function (11) used in the industrial design settings as our design target.

$$F_{sys} = \frac{6000000(s + 1000000)}{s^3 + 50000s^2 + 1000000000s + 3000000000000} \quad (11)$$

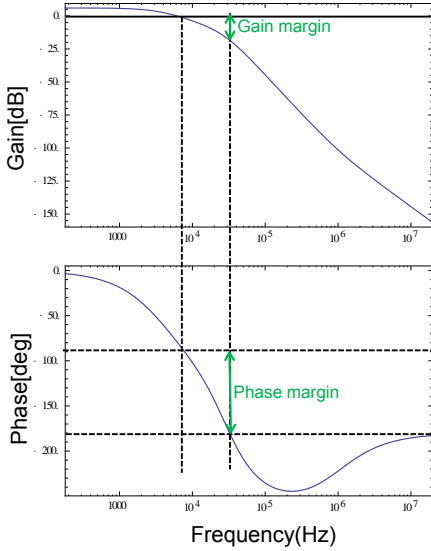


Fig.6: Bode Plot of our design target, according to (11)

When we assumed $T_{sys} = F_{sys}$, we got the following equation (12).

$$(U_2 s^2 + U_1 s + U_0)(s^3 + 50000s^2 + 1000000000s + 300000000000) - 6000000(D_4 s^4 + D_3 s^3 + D_2 s^2 + D_1 s + D_0)(s + 1000000) = 0 \quad (12)$$

Factoring out common terms, we are left with a fifth order polynomial in s with coefficients being integer polynomials in the variables $\{R_1, R_2, R_3, R_a, R_b, C_n, C\}$. To solve the equation (12), we need to set these coefficients to zero, then consequently we describe the set of equations (13):

$$\begin{aligned} 0 &= 37680000000000R_1R_a - 1880232000000000R_2 \\ &\quad + 37680000000000R_1R_b + 1887768000000000R_2R_b \\ 0 &= 376800000R_a + 6000037680000R_1R_a - 279962320000R_2R_a \\ &\quad + 37680000000000CR_1R_2R_a + 3768000000000CR_1R_3R_a \\ &\quad - 18802320000000000CR_2R_3R_a + 376800000R_b \\ &\quad + 6000037680000R_1R_b - 242282320000R_2R_b \\ &\quad + 376800000000000CR_1R_2R_b + 18877680000000000CR_2R_3R_b \\ &\quad + 376800000000000C_nR_1R_aR_b + 1887768000000000C_nR_2R_aR_b \\ 0 &= \frac{300001884R_a + 300001884R_b}{5} + 6000000R_1R_a \\ &\quad - 308000000R_2R_a + 376800000CR_2R_a + 6000037680000CR_1R_2R_a \\ &\quad + 3768000000CR_3R_a + 6000037680000CR_1R_3R_a \\ &\quad - 279962320000CR_2R_3R_a + 6000000R_1R_b - 308000000R_2R_b \\ &\quad + 376800000CR_2R_b + 6000037680000CR_1R_2R_b \\ &\quad + 376800000CR_3R_b + 6000037680000CR_1R_3R_b \\ &\quad - 242282320000CR_2R_aR_b + 37680000000000C_nR_1R_2R_aR_b \\ &\quad + 37680000000000C_nR_1R_3R_aR_b + 18877680000000000C_nR_2R_3R_aR_b \end{aligned}$$

$$\begin{aligned} 0 &= \frac{300001884CR_2R_a + 300001884CR_3R_a}{5} + \frac{6000000CR_1R_2R_a + 60R_a}{5} \\ &\quad + \frac{300001884CR_2R_b + 300001884CR_3R_b + 300001884C_nR_aR_b}{5} \\ &\quad - 6280R_2R_a + 6000000CR_1R_3R_a - 308000000CR_2R_3R_a + 60R_b \\ &\quad - 6280R_2R_b + 6000000CR_1R_2R_a + 6000000CR_1R_3R_b \\ &\quad - 308000000CR_2R_3R_b + 6000000C_nR_1R_aR_b - 308000000C_nR_2R_aR_b \\ &\quad + 376800000CR_2R_a + 6000037680000CR_1R_2R_a \\ &\quad + 376800000C_nR_2R_aR_b + 6000037680000C_nR_1R_2R_aR_b \\ &\quad + 376800000C_nR_3R_aR_b + 6000037680000C_nR_1R_3R_aR_b \\ &\quad - 242282320000C_nR_2R_3R_aR_b \\ 0 &= 60CR_2R_a + 60CR_3R_a - 6280CR_2R_3R_a + 60CR_2R_b + 60CR_3R_b \\ &\quad - 6280CR_2R_3R_b + 60C_nR_aR_b - 6280C_nR_2R_aR_b \\ &\quad + \frac{300001884C_nR_2R_aR_b + 300001884C_nR_3R_aR_b}{5} \\ &\quad + 6000000C_nR_1R_3R_aR_b - 308000000C_nR_2R_3R_aR_b \\ 0 &= 60C_nR_2R_aR_b + 60C_nR_3R_aR_b - 6280C_nR_2R_3R_aR_b \end{aligned} \quad (13)$$

If we use Mathematica's NSolve-procedure as a numerical engine, we can get inappropriate solution for six nonlinear equations (13). Indeed the two numerical solutions were $\{R_1=0, R_2=0, R_3=0, C=0, R_a=2.41411e9, R_b=2.40405e9, C_n=0\}$ or $\{R_1=0, R_2=11.152, R_3=0, C=0, R_a=4.29255e7, R_b=4.27468e7, C_n=0\}$. These circuit parameters are not adequate as design task.

Instead of such an approach, we apply Mathematica's Solve-procedure to calculate symbolic solutions. Then we found four or five exact solutions and one or two symbolic solutions as function of R_a , voicing the explicit caveat: "Equations may not give solutions for all 'solve' variables". We have obtained reasonable and exact solutions of a total of four, but we omitted the other three. Now one of the solutions is shown in (14), and we omit other solutions.

$$\begin{aligned} R_1 &= \frac{94350069732743 + 3\sqrt{921191000538820354571811561}}{384801505000000} \\ R_2 &= \frac{3(94350069732743 + 3\sqrt{921191000538820354571811561})}{49875592162604} \\ R_3 &= \frac{3(94350069732743 + 3\sqrt{921191000538820354571811561})}{314(94191230267257) + 3\sqrt{921191000538820354571811561}} \\ R_b &= \frac{917194231517257}{921029518482743} R_a \\ C_n &= 0 \\ C &= \frac{3753664980267257 + 3\sqrt{921191000538820354571811561}}{3676447500000000000} \end{aligned} \quad (14)$$

Then we numerically calculate solutions of above or other for the exact solutions:

Solution 1) $\{R_1=0.481816, R_2=11.152, R_3=0.00956233, C=0.000104577, C_n=0, R_b=0.995836 R_a\}$

And other solutions are:

Solution 2) $\{R_1=0.00856723, R_2=0.198294, R_3=0.0100378, C=0.0000996237, C_n=0, R_b=0.995836 R_a\}$

Solution 3) $\{R_1=0.00793284, R_2=0.198034, R_3=0.0100384, C=0.000099617, C_n=2.00417e-6/R_a, R_b=0.995848 R_a\}$

Solution 4) $\{R_1=0.48245, R_2=12.0438, R_3=0.00956233, C=0.000104584, C_n=2.00417e-6/R_a, R_b=0.995848 R_a\}$.

We consider the implications about these solutions. The symbolic value R_b, C_n as function of R_a are circuit elements constructed the feedback circuit described in Fig.4. And $C_n=0$ of solution1) and solution2) represents that the coefficient of the second term of numerator and the forth term of denominator in the polynomial T_{sys} are zero. That is a loss of generality from the perspective of the circuit. But, we substitute a value R_a to obtain suitable parameters. For example, we assume that $R_a=20000$, then $R_b=199167.2$ from solution 2).

Another approach to get the solution of following equation $T_{sys} = F_{sys}$ is that we multiply $ax+b$ to equalize the degree of the numerator and denominator of F_{sys} and T_{sys} . Namely,

$$F_{sys}' = \frac{6000000(s+1000000)(as+b)}{(s^2+50000s^2+1000000000s+300000000000)(as+b)} \quad (15)$$

Then we solved the following equation, $T_{sys} = F_{sys}'$. Here of the numerator of T_{sys} is equal to the numerator of F_{sys}' , and the denominator of T_{sys} is equal to the denominator of F_{sys}' . As the results, we can get the solution 3), where $R_a=2.4172688377894015e9$.

In brief, we got the solution 4) without the loss of generality.

5 Conclusions

We discussed a method of analog circuit design with symbolic analysis. For a power supply stabilization loop circuit which is a main component of power supply IC, we derived a transfer function of the circuit characteristics. We gave a circuit design target in the form of transfer function. And we formulated as a problem to solve a system of nonlinear equations in order to decide the most suitable set of constants of circuit elements which satisfies this target by using symbolic algebraic manipulation so-called Mathematica. As a result of our investigation, we reveal that their equations are able to be solved with not the numerical method but symbolic algebra method and there are plural solutions which have important sense from circuit-like point of view.

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